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Unimodular relativity and cosmological constant

David R. Finkelstein, Andrei A. Galiautdinov, and James E. Baugh
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0430

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Unimodular relativity is a theory of gravity and space–time with a fixed absolute space–time volume element, the *modulus*, which we suppose is proportional to the number of microscopic modules in that volume element. In general relativity an arbitrary fixed measure can be imposed as a gauge condition, while in unimodular relativity it is determined by the events in the volume. Since this seems to break general covariance, some have suggested that it permits a nonzero covariant divergence of the material stress-energy tensor and a variable cosmological ''constant.'' In Lagrangian unimodular relativity, however, even with higher derivatives of the gravitational field in the dynamics, the usual covariant continuity holds and the cosmological constant is still a constant of integration of the gravitational field equations. © *2001 American Institute of Physics*. [DOI: 10.1063/1.1328077]

I. INTRODUCTION TO UNIMODULAR RELATIVITY

Unimodular relativity is an alternative theory of gravity considered by Einstein in 1919[1] without a Lagrangian and put into Lagrangian form by Anderson and Finkelstein.[2] The space–time of unimodular relativity is a measure manifold, a manifold provided by nature with a fixed absolute physical measure field $\mu(x)$ to be found by direct measurement, subject to no dynamical development. The sole structural variable is a conformal metric tensor $f_{\mu\nu}$, subject to dynamical equations. The measure of a space–time region may be regarded as indirectly counting the modules of which it is composed, in the way that the volume of a lake indirectly counts its water molecules. Both space–time measure and liquid measure indicate a modular structure below the limit of resolution of the present measuring instruments.

The conformal metric field $f_{\mu\nu}(x)$ is a symmetric relative tensor of weight 1/2, signature 1–3, and determinant -1 in all coordinate systems, with nine independent components, operationally defined by the system of light paths, whose tangent vectors dx^μ obey $f_{\mu\nu}dx^\mu dx^\nu=0$.

The unimodular space–time structure also defines a metric tensor

$$g_{\mu\nu}=\sqrt{\mu}f_{\mu\nu}(x), \tag{1}$$

but the determinant

$$-g:=\det g_{\mu\nu}=-\mu^2 \tag{2}$$

is not a dynamical variable. The conformal metric f is the sole gravitational variable of unimodular relativity.

We assume that the metric tensor field $g_{\mu\nu}$ found by measuring the proper times $d\tau^2=g_{\mu\nu}dx^\mu dx^\nu$ for a sufficiently fine network of intervals dx^μ, also determines the measure field μ by the usual relation (2).

Once the measure μ has been experimentally determined it establishes a class of admissible metrics obeying (2). Metrics violating (2) are unphysical according to unimodular relativity.

The variable of general relativity is a compound of a light-cone field f and a measure field μ, and the group of general relativity is a nonsimple group of diffeomorphisms, with an invariant subgroup of unimodular coordinate transformations, those with Jacobian $\det(\partial x'/\partial x)=1$. Unimodular relativity has a simple group and a simple variable.

Originally we proposed unimodular relativity because there is indeed an experimental atomic standard of length near each point of space–time, not built into general relativity.[2] This suggests that the macroscopic structure of space–time is a smoothed description of an underlying atomic space–time microstructure, which seems necessary for other reasons.

Since the actual value of the cosmological constant is so finely tuned, it is natural to attempt to derive its value from physical principles. A theory in which it is variable would be a useful starting point for any such attempt. Recently the difference in symmetry between unimodular and general relativity led some to hope that they might differ on the constancy of this parameter.[3]

On the other hand, many authors have already argued that the difference is only a gauge condition, which has no physical consequences.[4] However, some authors do not share this point of view. In particular, van der Bij *et al*.[5] stated very clearly the physical difference between the usual formulation of gravity and the unimodular theory. Also, an interesting and somewhat alternative approach is presented in Ref. 6.

We examine the gauge-condition argument more carefully here. In its usual form it omits several relevant features special to this problem. Usually gauge conditions are applied to Lagrangians that are already physically well defined in their absence; the unimodularity condition is not a gauge condition of this kind.

One should also take into account the possibility of higher-order derivatives in the gravitational equations, of the kind that might arise from renormalization in some hypothetical quantum field theory of gravity.

We show here that any gravitational theory of classical unimodular relativity with a Lagrangian density that is invariant under the unimodular coordinate group is equivalent in its experimental predictions to a theory of classical general relativity. Higher-order corrections do not disturb this equivalence.

II. THE METRIC TENSOR OF UNIMODULAR RELATIVITY

In deriving the field equations from a variational principle (on which our approach is based), the measure μ is not varied but is treated as if it were a fixed external field. This disturbs general covariance. The law of nature may take a simpler form in *unimodular coordinates*, where $\mu(x) \equiv 1$. Unimodular coordinate systems are related by unimodular transformations.

Let R be the Riemann scalar computed from the metric tensor $g_{\mu\nu}$ of (1). Let L_M be the Lagrangian density of the matter field in the presence of $g_{\mu\nu}=\sqrt{\mu}f_{\mu\nu}$. Then

$$S=\int d^4x\left(\frac{\Gamma}{2}R+L_M\right) \tag{3}$$

is a possible action functional for unimodular relativity in a unimodular coordinate system. The constant $\Gamma=1/4\pi G$ is the inverse rationalized gravitational constant, the reciprocal square of the rationalized Planck length, in units with $\hbar=c=1$.

In unimodular relativity, there is initially no way to vary all 10 components of $g_{\mu\nu}$ independently. The action is in principle defined only for $g=\mu^2$. Only derivatives with respect to the nine-dimensional conformal metric field f are defined.

A cosmological constant term $\Lambda\sqrt{g}$ in the action function would be an ineffectual additive constant since $\sqrt{g}=\mu$ is not varied.

This action can be transformed to any other coordinate system under the general diffeomorphism group, but is not generally invariant in functional form, since the fixed measure μ sets an absolute scale at each event.

The derivative with respect to the conformal metric f requires special care. Since infinitesimal variations δf are subject to the unimodular condition (2), they obey

$$f^{\mu\nu}\delta f_{\mu\nu}=0. \tag{4}$$

If $W:\mathcal{F}\to\mathcal{W}$, $f\mapsto W(f)$ is a functional from the function manifold $\mathcal{F}$ of conformal metrics on a region $\mathcal{R}$ to some value-manifold $\mathcal{W}$, we define the functional derivative $W_f=\delta W/\delta f$ as the linear operator

$$W_f: d\mathcal{F}\to\mathcal{W} \tag{5}$$

such that for any $\delta f\in d\mathcal{F}$ vanishing on the boundary $\partial\mathcal{R}$ the tangent space to $\mathcal{F}$ at f,

$$\delta W = W_f\cdot\delta f = \int d^4x\,\frac{\delta W}{\delta f_{\mu\nu}(x)}\,\delta f_{\mu\nu}(x). \tag{6}$$

Then the dynamical equation that follows from the action principle for any space–time region $\mathcal{R}$ is

$$\delta S(\mathcal{R}) = \int_{\partial\mathcal{R}} d\sigma_\mu\,\pi^\mu\cdot\delta f \tag{7}$$

with a boundary term that is linear in δf on the boundary $\partial\mathcal{R}$ and vanishes for variations that vanish on the boundary. The tensor field π^μ canonically conjugate to $f(x)$ is defined by these relations.

III. FIELD EQUATIONS AND THE COSMOLOGICAL CONSTANT

It is often inconvenient to work with a field variable subject to non-linear conditions like the conformal metric. One may reformulate unimodular relativity with an unconstrained variable $g_{\mu\nu}(x)$ and take the unimodular condition (2) into account through Lagrange's method of undetermined multipliers.

To do this, however, we must give values to the Lagrangian density for metric tensors that have $g\neq\mu$[22,4,7] and hence are unphysical. One way to do this is to replace μ by $\sqrt{g}$ and $f_{\mu\nu}$ by $g_{\mu\nu}/g^{1/4}$ at all their ''appearances'' in the action S, so that S is defined in a 10-dimensional neighborhood $\mathcal{F}'\supset\mathcal{F}$ in a smooth way consistent with the values on $\mathcal{F}$. Then in addition one adds a Lagrangian-multiplier term expressing the unimodular condition. We call the resulting extension of S to $\mathcal{F}'$ the *extended action function* S'.

But this prescription is ambiguous, since it depends on ''appearances,'' on how S is written, on matters of notation. The extended action S' is arbitrary up to a correction term $\Delta_M S$ depending on the matter variables and the metric $g_{\mu\nu}$, subject only to the condition that $\Delta_M L$ and its derivative with respect to λ vanish in the unimodular sector (2):

$$S' = S+\Delta_M S+\int d^4x\sqrt{g}\lambda(x)\left[\frac{\mu}{\sqrt{g(x)}}-1\right], \tag{8}$$

$\Delta_M S=\int d^4x\sqrt{g}\Delta_M L$ is the *unimodular ambiguity* in the action. No physical results may depend on the choice of $\Delta_M L$, and so no physical experiments can determine $\Delta_M L$.

In mechanical theories sometimes we impose a constraint and thereby reduce an already well-defined system to a system of lower dimensionality. For example, we reduce a free particle to a spherical pendulum by constraining it to a sphere. Then there is a well-defined unconstrained Lagrangian, found by removing the constraint, and the ambiguity $\Delta_M L$ does not arise.

But according to unimodular relativity we have no way of actually removing the unimodular condition (2). In this sense it is not a constraint, so we call it a condition. While the proper time $d\tau$ of a coordinate interval dx^μ at x depends on the gravitational field at x, each coordinate cell d^4x at x comes with its own intrinsic measure $\mu(x)d^4x$, independent of gravity. The unimodular ambiguity $\Delta_M L$ acknowledges that as a matter of principle we cannot know how the system would evolve absent the unimodular condition.

The admissible infinitesimal variations df within the neghborhood $\mathcal{F}'$ are those that obey (4) for all $x \in \mathcal{R}$. That is, they all belong to the null space of the inverse conformal metric tensor $f^{\mu\nu}$.

The action principle states that for any physical field $f=(f_{\mu\nu}(x))$, and for any variations $\delta f=(\delta f_{\mu\nu}(x))$ about f,

$$S_f \cdot \delta f := \int_{\mathcal{R}} d^4x\, S_f(x)\, \delta f(x) = \int_{\partial\mathcal{R}} S^{\mu\nu}\delta f_{\mu\nu}. \tag{9}$$

That is, any vector $\delta f(x)$ in the null space of $f^{-1}(x)$ is in the null space of $S_f(x)$. It follows that $S_f(x)$ is in the ray of $f^{-1}(x)$:

$$\frac{\delta S}{\delta f_{\mu\nu}(x)} = \lambda(x) f^{\mu\nu}(x). \tag{10}$$

The multiplier $\lambda(x)$ is then fixed by the unimodular condition.

This implies that the augmented action (8) is stationary up to boundary terms when we vary $\lambda(x)$ and the 10 components $g_{\mu\nu}(x)$ independently.

The unimodular condition makes the stress tensor ambiguous as well as the dynamical equations. In unimodular relativity the general relativistic concept of the stress tensor

$$T^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta(\sqrt{g} L_M)}{\delta g_{\mu\nu}} \tag{11}$$

has no principled meaning at first, since it involves breaking the unimodular condition, nor has the statement of covariant continuity, $T^{\mu\nu}{}_{;\nu}=0$.

We may suppose that $\Delta_M L$ has the form

$$\Delta_M L = \left[\frac{\mu(x)}{\sqrt{g(x)}} - 1 \right] l_M, \tag{12}$$

where l_M is any function of the matter variables and $g_{\mu\nu}$. We write $L'_M := L_M + \Delta_M L$ for the sum.

From any general-relativity action principle S we obtain in this way an ambiguous unimodular-relativity action principle

$$S' = \int \sqrt{g(x)}\, d^4x \left\{ \frac{\Gamma}{2} R(x) + \frac{\Gamma}{2} \lambda(x) \left[\frac{\mu(x)}{\sqrt{g(x)}} - 1 \right] + L_M + \Delta_M L \right\}. \tag{13}$$

The second term in S' expresses the unimodular condition and breaks general covariance. The fourth term expresses the unimodular ambiguity. We vary the 10 $g_{\mu\nu}(x)$ and the Lagrange multiplier $\lambda(x)$ independently. We have written the λ term in a form that makes λ a scalar field under the general group.

Variation of λ in S' recovers the condition

$$\sqrt{g(x)} = \mu(x). \tag{14}$$

The unimodular ambiguity $\Delta_M L$ does not affect this result, since it vanishes when the unimodular condition holds.

Variation of $g_{\mu\nu}$ gives the equation of motion

$$\delta S' = \int \sqrt{g}\, d^4x \{\Gamma(\tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu}) + \tfrac{1}{2}\Gamma\lambda g^{\mu\nu}\}\delta g_{\mu\nu} + \int \sqrt{g}\, d^4x \left\{ \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{g} L_M + \sqrt{g}\Delta_M L) \right\} \delta g_{\mu\nu} = 0, \tag{15}$$

or

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R - \tfrac{1}{2}\lambda g^{\mu\nu} = 8\pi G T'^{\mu\nu}, \tag{16}$$

where $G := 1/\Gamma$ is a rationalized gravitational coupling strength and

$$T'^{\mu\nu} := \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{g} L_M + \sqrt{g}\Delta_M L). \tag{17}$$

Tracing (16) gives the Lagrange multiplier:

$$\lambda = -\tfrac{1}{2}(8\pi G T' + R). \tag{18}$$

The field equation then simplifies to

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = 8\pi G T'^{\mu\nu} - \tfrac{1}{4} g^{\mu\nu}(8\pi G T' + R), \tag{19}$$

or equivalently

$$R_{\mu\nu} - \tfrac{1}{4} R g_{\mu\nu} = 8\pi G T'^{\mathrm{T}}_{\mu\nu}, \tag{20}$$

where

$$T'^{\mathrm{T}}_{\mu\nu} := T'_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} T' \tag{21}$$

is the traceless (part of the) stress tensor, or the *sess tensor* (note that the tr has been removed). The covariant divergence of (19) is

$$8\pi G T'^{\mu\nu}{}_{;\nu} = \tfrac{1}{4} g^{\mu\nu}(8\pi G T' + R)_{,\nu} \equiv -\tfrac{1}{2} g^{\mu\nu}\lambda, \tag{22}$$

which was suggested as a ''modified covariant divergence law'' by Tiwari.[3]

In general relativity, general invariance of the action function implies that the covariant divergence $T'^{\mu\nu}{}_{;\nu}$ vanishes in virtue of the field equations for matter.[8,9] Then it follows that

$$-\tfrac{1}{2}(8\pi G T' + R) = \lambda = \text{constant}. \tag{23}$$

If the stress tensor of unimodular relativity were covariantly continuous too, then the undetermined multiplier λ could be identified with a cosmological constant Λ, though now a constant of the motion determined by the initial data, rather than a fixed absolute constant as supposed in general relativity.

In unimodular relativity however, $T'^{\mu\nu}$ is not covariantly continuous because the action S' is not generally covariant, which seems to justify Tiwari's suggestion. $T'^{\mu\nu}$ is ambiguous by an additive term

$$\Delta T^{\mu\nu} = -\tfrac{1}{2} g^{\mu\nu} l_M \tag{24}$$

and its trace is correspondingly ambiguous by

$$\Delta T = -2l_{\mathrm{M}}. \tag{25}$$

However substitution of these expressions into (20) immediately leads to

$$R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu} = 8\pi G T^{\mathrm{T}}_{\mu\nu}, \tag{26}$$

with

$$T^{\mathrm{T}}_{\mu\nu} := T_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}T, \tag{27}$$

where $T_{\mu\nu}$ is the usual covariantly continuous stress-energy tensor (11) of the matter field. The gravitational field equations *do not* depend on the ambiguity in the matter field Lagrangian. The cosmological constant again arises as a constant of integration.

Einstein's law of gravity is a second-order differential equation for the metric field. In quantum field theories higher-order derivatives arise from renormalization. One might hope that in higher-order theories, the unimodular relativity differs in content from general relativity, and allows the cosmological constant to vary. In such theories the cosmological constant may be defined as $\lambda = S(h)/\mu$ where $S(h)$ is the value of the gravitational action density for the case of the Minkowski metric.

It easily follows from the generalized (contracted) Bianchi identities of the higher order theory (see, for example, Refs. 10 and 11) that even in higher-order unimodular relativity, where the gravitational part of the action is an arbitrary generally invariant functional of the curvature scalar, the cosmological constant also arises as the constant of integration.

IV. CONCLUSIONS

We have shown that in Lagrangian unimodular relativity the usual covariant continuity equation holds for the source stress tensor, and the cosmological constant is a constant of integration of the gravitational field equations. Higher derivatives of the gravitational field may appear in the Lagrangian without disturbing these conclusions. The essential point is that the stress tensor have no covariant divergence. This follows from either unimodular or general covariance.

There are several reasons not to be quite certain that these classical conclusions will still hold in the quantum domain.

The fact remains that unimodular relativity forces us to allow many values as possibilities for the cosmologcal constant, while general relativity fixes on one value. In quantum theories, possibilities affect actualities. In a quantum theory of sufficient scope, these many possibilities for the cosmological constant of unimodular relativity might influence the actual experimental situation.[7]

Furthermore, quantum field theories often lack symmetries and conservation laws present in the classical Lagrangian from which they stem, due to divergences inherent in the limiting process used to define the quantum theory. This results in quantum anomalies, for example. Similar effects may permit the cosmological constant—the vacuum rest-energy-density—to vary in some quantum version of unimodular relativity.

Perhaps the most basic weakness in the deduction is the postulate of strong locality. This is implicit in general covariance and is required to deduce the covariant continuity of stress. If there is a fundamental quantum time, a limit to locality, as some suggest, then the cosmological constant can vary.

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[1] A. Einstein, Sitzungsber. K. Preuss. Akad. Wiss., Pt. 1, 433 (1919). English translation in H.A. Lorentz *et al*., *The Principle of Relativity* (Dover, New York, 1952).
[2] J. Anderson and D. Finkelstein, Am. J. Phys. **39**, 901 (1971).
[3] S.C. Tiwari, J. Math. Phys. **34**, 2465 (1993).
[4] F. Wilczek, Phys. Rep. **104**, 111 (1984); A. Zee, in *High Energy Physics*, Proceedings of the 20th Annual Orbis Scientiae, Coral Gables, 1983, edited by B. Kursunoglu, S.C. Mintz, and A. Perlmutter (Plenum, New York, 1985); W. Buchmüller and N. Dragon, Phys. Lett. B **207**, 292 (1988); W.G. Unruh, Phys. Rev. D **40**, 1048 (1988); M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989); L. Bombelli *et al*., Phys. Rev. D **44**, 2589 (1990).
[5] J.J. van der Bij *et al*., Physica A **116**, 307 (1982).
[6] R.D. Sorkin, Int. J. Theor. Phys. **33**, 523 (1994); ''Problems with Causality in the Sum-over-histories Framework for Quantum Mechanics,'' in *Conceptual Problems of Quantum Gravity*, edited by A. Ashtekar and J. Stachel, Proceedings of the Conference Held at Osgood Hill, MA, May 1988 (Birkhäuser, Boston, MA, 1991), P. 217; Int. J. Theor. Phys. **36**, 2759 (1997); also LANL gr-qc/9706002; A. Daughton, J. Louko, and R.D. Sorkin, Phys. Rev. D **58**, 084008 (1998); also LANL gr-qc/9805101.
[7] Y.J. Ng and H. van Dam, J. Math. Phys. **32**, 1337 (1991), LANL hep-th/9911102.
[8] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th revised English ed. (Pergamon, New York, 1975), p. 270
[9] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[10] L. Querella, Doctoral dissertation, Universite de Liege, 1998.
[11] R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984), Appendix E.