Post-Newtonian celestial mechanics in scalar-tensor cosmology

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Applying the recently developed dynamical perturbation formalism on cosmological background to scalar-tensor theory, we provide a solid theoretical basis and a rigorous justification for phenomenological models of orbital dynamics that are currently used to interpret experimental measurements of the time-dependent gravitational constant. We derive the field equations for the scalar-tensor perturbations and study their gauge freedom associated with the cosmological expansion. We find a new gauge eliminating a prohibitive number of gauge modes in the field equations and significantly simplifying post-Newtonian equations of motion for localized astronomical systems in the universe with a time-dependent gravitational constant. We identify several new post-Newtonian terms and calculate their effect on secular cosmological evolution of the osculating orbital elements.

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I. INTRODUCTION

Alternative theories of gravity—the competitors to Einstein's general theory of relativity—have been the subject of numerous investigations for almost a hundred years. One such theory is the scalar-tensor theory [1–3], an outgrowth of theories by Jordan [4] and Brans and Dicke [5], in which, in addition to the metric tensor, the gravitational field is described by the fundamental scalar field, ϕ. This so-called Brans-Dicke (BD) field has a broader meaning than the scalar field of standard general relativity, where it is present only in the stress-energy tensor and generates curvature via Einstein's field equations. In scalar-tensor theory, apart from having its own stress-energy tensor, the BD field appears explicitly in the Lagrangian through direct coupling to the curvature scalar, which makes Newton's gravitational constant variable, $G \propto 1/\phi$, and gives rise to additional terms in Einstein's field equations, presumably, with important observational consequences.

A typical approach to deriving such consequences is to assume that the cosmological evolution of the scalar field affects the value of the gravitational constant, making it time dependent, much as in the earlier proposals by Dirac [6]. One then postulates a linear time dependence,

$$G(t) = G_0[1 + (\dot{G}_0/G_0)(t - t_0)], \tag{1}$$

with G_0 representing the value of G at reference epoch, say, $t_0 = \text{J2000}$ (which we set to zero, for convenience), and uses it in the equation of motion for the gravitational probe moving in a spherically symmetric gravitational field of a pointlike mass M [7,8],

$$\ddot{\boldsymbol{r}} = -\frac{G(t)M\boldsymbol{r}}{r^3} + \boldsymbol{F}_{\text{Newtonian}} + \boldsymbol{F}_{\text{relativistic}}, \tag{2}$$

where the first term on the right-hand side of (2) represents what is known as the Gyldén-Meshcherskii problem [9,10], the term $\boldsymbol{F}_{\text{Newtonian}}$ includes additional Newtonian corrections, such as the influence of the other planets, and $\boldsymbol{F}_{\text{relativistic}}$ includes the relativistic terms present in the Einstein-Infeld-Hoffman equations [11]. The so-called linear trend, $\dot{G}_0/G_0$, is then estimated from the astronomical observations. The Lunar Laser Ranging experiment based on the 44 years of data [12] and the Mars Reconnaissance Orbiter experiment [13] give the most stringent upper limits on the variability of G, $\dot{G}_0/G_0 = (1.4 \pm 1.5) \times 10^{-13}\ \text{yr}^{-1}$ and $\dot{G}_0/G_0 = (0.1 \pm 1.6) \times 10^{-13}\ \text{yr}^{-1}$, correspondingly.

The above phenomenological approach can certainly be improved. We are particularly interested in determining the relativistic terms that come from a careful analysis of the scalar-tensor theory of an expanding universe. That can be done with the help of the dynamical perturbation theory of curved spacetime manifolds recently developed in Refs. [14,15], which provides a rigorous method of calculating the gravitational fields of perturbations whose density contrast significantly exceeds the average density of the universe, such as in the case of a localized gravitational system placed on the cosmological background. The field variables of the theory are then naturally separated into two parts: the background part, whose dynamics is fully determined by the spherically symmetric Freedman solution of the Brans-Dicke theory, and perturbations, whose evolution is governed by the field equations derived on the basis of the properly formulated variational procedure applied to the Brans-Dicke action functional.

It is the perturbations of the background metric and the scalar field that we are interested in finding. They determine the effective gravitational force, from which the post-Newtonian terms in the equations of motion can be deduced, thus improving on Eq. (2).

II. LAGRANGIAN, FIELD EQUATIONS, AND METRIC

We take the localized system to be a planetary system or a binary pulsar and the background manifold to be the spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) solution of the scalar-tensor theory. The background metric in the isotropic conformal coordinates, $x^\mu \equiv (c\eta, x^i)$, $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$, thus has the form

$$\bar{g}_{\mu\nu} = a^2(\eta)\mathrm{f}_{\mu\nu}, \qquad \mathrm{f}_{\mu\nu} \equiv \mathrm{diag}(-1, 1, 1, 1), \tag{3}$$

where c is the speed of light, $a(\eta)$ is the cosmological scale factor, and η is the conformal time that is related to the standard cosmological time, $t_{\rm H}$, measured by freely falling Hubble observers via $dt_{\rm H} = a(\eta)d\eta$. Additionally, in terms of η, the conformal Hubble constant is defined by

$$\mathcal{H} \equiv (1/a)(da/d\eta). \tag{4}$$

The full gravitational system is then described by the Lagrangian, $\mathcal{L} = \mathcal{L}^{\rm st} + \mathcal{L}^{\rm bgc} + \mathcal{L}^{\rm p}$, where

$$\mathcal{L}^{\rm st} = \frac{\sqrt{-g}c^3}{16\pi}\left(-\phi R + \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\right) \tag{5}$$

is the Lagrangian of the scalar-tensor theory, R is the Ricci scalar curvature, ϕ is the BD field, $\phi_{,\alpha} \equiv \partial_\alpha\phi \equiv \partial\phi/\partial x^\alpha$, ω is the BD coupling parameter (assumed to be a function of ϕ), $g^{\alpha\beta}$ is the (inverse) metric, $g \equiv \det(g_{\mu\nu})$, $\mathcal{L}^{\rm bgc}$ is the Lagrangian of the background content of the expanding universe (mainly dark matter and dark energy, but also includes background baryonic matter), and $\mathcal{L}^{\rm p}$ is the perturbing Lagrangian of the localized system. The $\mathcal{L}^{\rm p}$ is assumed to be independent of ϕ, which corresponds to the requirement that the geodesic motion of material objects is governed by the metric alone without any direct influence of the BD field.

In accordance with the dynamical perturbation formalism, we write the full metric and the BD field as the sums,

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(\eta) + \varkappa_{\mu\nu}(x), \qquad \phi(x) = \bar{\phi}(\eta) + \varphi(x), \tag{6}$$

of their background parts, $\bar{g}_{\mu\nu}$ and $\bar{\phi}$, and perturbations, $\varkappa_{\mu\nu}$ and φ. We also introduce the contravariant metric density, $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta}$, its background value, $\bar{\mathfrak{g}}^{\alpha\beta} \equiv \sqrt{-\bar{g}}\bar{g}^{\alpha\beta}$, and the perturbation, $\mathfrak{h}^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} - \bar{\mathfrak{g}}^{\alpha\beta}$, which is conveniently written in the form

$$\mathfrak{h}^{\alpha\beta} \equiv \sqrt{-\bar{g}}l^{\alpha\beta}. \tag{7}$$

We then take $\mathfrak{h}^{\alpha\beta}$ and φ to represent the dynamical variables of the theory and use the variational procedure of Refs. [14,15] to write down the linearized field equations for $\mathfrak{h}^{\alpha\beta}$ and φ,

$$\frac{-16\pi}{c^3\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\phi}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{\rm st}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^{\rm st}}{\delta\bar{\phi}}\right) = 0, \tag{8}$$

$$\frac{-16\pi}{c^3\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{\rm st}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^{\rm st}}{\delta\bar{\phi}}\right) = \frac{8\pi}{c^4}\Lambda_{\mu\nu}, \tag{9}$$

where $\delta\bar{\mathcal{L}}^{\rm st}/\delta\bar{\phi}$ and $\delta\bar{\mathcal{L}}^{\rm st}/\delta\bar{\mathfrak{g}}^{\rho\sigma}$ stand for the variational derivatives of $\bar{\mathcal{L}}^{\rm st} \equiv \mathcal{L}^{\rm st}(\bar{\mathfrak{g}}^{\alpha\beta}, \bar{\phi})$ with respect to the background field $\bar{\phi}$ and metric density $\bar{\mathfrak{g}}^{\rho\sigma}$, and

$$\Lambda_{\mu\nu} \equiv \frac{2c}{\sqrt{-\bar{g}}}\frac{\delta\mathcal{L}^{\rm p}}{\delta\bar{g}^{\mu\nu}} \tag{10}$$

is the stress-energy tensor of the localized gravitational system. Once Eqs. (8) and (9) are worked out, we can find $l^{\alpha\beta}$ and φ by solving these equations, and then, via

$$\varkappa_{\mu\nu} = -l_{\mu\nu} + (1/2)\bar{g}_{\mu\nu}l, \qquad l \equiv l^\alpha{}_\alpha, \tag{11}$$

find the full metric, $g_{\mu\nu}$. (Note that the background metric $\bar{g}_{\mu\nu}$ is used to raise and lower tensorial indices; covariant differentiation with respect to $\bar{g}_{\mu\nu}$ will be denoted with a vertical bar.)

Applying (8) and (9) to $\bar{\mathcal{L}}^{\rm st}$, and making the linearized Hubble approximation in which we ignore all terms containing $\mathcal{H}^2$, $d\mathcal{H}/d\eta$, $d^2\bar{\phi}/d\eta^2$, etc., we get the system of differential equations for scalar-tensor perturbations,

$$\varphi^{|\alpha}{}_{|\alpha} + \frac{2\omega'}{3+2\omega}\bar{\phi}^{|\alpha}\varphi_{|\alpha} + A^\alpha\bar{\phi}_{|\alpha} = \frac{8\pi\Lambda}{(3+2\omega)c^4}, \tag{12}$$

$$\begin{aligned}&(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}A^\alpha{}_{|\alpha} - A_{\mu|\nu} - A_{\nu|\mu}) + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}(l_{\mu\nu|\alpha} - l_{\alpha\mu|\nu} - l_{\alpha\nu|\mu}) - \bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(\frac{1}{2}l_{|\alpha} - \frac{2\omega}{\bar{\phi}}\varphi_{|\alpha}\right)\\ &+ \frac{\bar{\phi}_{|\mu}}{\bar{\phi}}\left(\frac{1}{2}l_{|\nu} - \frac{2\omega}{\bar{\phi}}\varphi_{|\nu}\right) + \frac{\bar{\phi}_{|\nu}}{\bar{\phi}}\left(\frac{1}{2}l_{|\mu} - \frac{2\omega}{\bar{\phi}}\varphi_{|\mu}\right) + 2\bar{g}_{\mu\nu}\frac{\bar{\phi}_{|\alpha}}{\bar{\phi}}A^\alpha + \frac{2}{\bar{\phi}}(\bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha} - \varphi_{|\mu\nu}) = \frac{16\pi}{\bar{\phi}c^4}\Lambda_{\mu\nu},\end{aligned} \tag{13}$$

where $A^\alpha \equiv l^{\alpha\beta}{}_{|\beta}$, $\omega \equiv \omega(\bar\phi)$, $\omega' \equiv d\omega(\bar\phi)/d\bar\phi$, and $\Lambda \equiv \bar g^{\mu\nu}\Lambda_{\mu\nu}$. Equations (12) and (13) admit an enormous number of gauge modes most of which can be eliminated if we impose the gauge condition [here, $\bar u^\alpha = (1/a, 0, 0, 0)$ is the velocity of the Hubble flow],

$$A^\alpha = -\frac{2\mathcal{H}}{ac} l^{\alpha\beta}\bar u_\beta - l^{\alpha\beta}\frac{\bar\phi_{|\beta}}{\bar\phi} + \frac{\bar\phi^{|\alpha}}{\bar\phi}\left(\frac{l}{2} - \frac{2\omega}{\bar\phi}\varphi\right) - \frac{\varphi^{|\alpha}}{\bar\phi} - \frac{2\mathcal{H}}{ac}\frac{\varphi}{\bar\phi}\bar u^\alpha - \frac{\bar\phi^{|\alpha}\varphi}{\bar\phi^2}, \tag{14}$$

which generalizes the gauges used in Refs. [5] and [14]. Using (14) and rewriting everything in the isotropic conformal coordinates, we arrive at the wave equations for perturbations,

$$\Box\varphi + \frac{2}{c}\left(-\mathcal{H} + \frac{\mathcal{F}}{2} - \frac{\omega'\bar\phi\mathcal{F}}{3+2\omega}\right)\varphi_{,0} = \frac{8\pi \mathsf{f}^{\alpha\beta}\Lambda_{\alpha\beta}}{(3+2\omega)c^4}, \tag{15}$$

$$\Box Q_{\mu\nu} + \frac{2}{c}\left(\mathcal{H} - \frac{\mathcal{F}}{2}\right)Q_{\mu\nu,0} = \frac{16\pi a^2}{\bar\phi c^4}\Lambda_{\mu\nu}, \tag{16}$$

where $\Box\varphi \equiv \mathsf{f}^{\alpha\beta}\varphi_{,\alpha\beta}$ and $\Box Q_{\mu\nu} \equiv \mathsf{f}^{\alpha\beta}Q_{\mu\nu,\alpha\beta}$. In the above, we introduced an auxiliary gravitational variable

$$Q_{\mu\nu} \equiv l_{\mu\nu} + \bar g_{\mu\nu}\varphi/\bar\phi, \tag{17}$$

and defined

$$\mathcal{F} \equiv (1/\bar\phi)(d\bar\phi/d\eta). \tag{18}$$

Equations (15) and (16) have the general form

$$\Box Q + (2/c)\mathcal{B}Q_{,0} = 4\pi a^2\mathcal{T}, \tag{19}$$

with $\mathcal{B}(\eta) \sim \mathcal{O}(\mathcal{H})$, $d\mathcal{B}(\eta)/d\eta \sim \mathcal{O}(\mathcal{H}^2)$. This can be solved by introducing two new functions, $b = b(\eta)$ and $q = q(\eta, x^i)$, such that $Q = b^2 q$, with $db/d\eta = \mathcal{B}b$. Noticing that, in the linear Hubble approximation,

$$\Box Q + (2/c)\mathcal{B}Q_{,0} = b\Box(bq), \tag{20}$$

we get the equation

$$\Box(bq) = 4\pi\frac{a^2\mathcal{T}}{b}, \tag{21}$$

whose retarded solution is given by the volume integral,

$$q(\eta, \mathbf{x}) = -\frac{1}{b(\eta)}\int \frac{a^2(\eta')}{b(\eta')}\frac{\mathcal{T}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{22}$$

with $\eta' = \eta - |\mathbf{x}-\mathbf{x}'|/c$ being the retarded time. The corresponding solution to (19) is then

$$Q(\eta, \mathbf{x}) = -b(\eta)\int \frac{a^2(\eta')}{b(\eta')}\frac{\mathcal{T}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \tag{23}$$

Applying (23) to (15) and (19) we get

$$\varphi(\eta, \mathbf{x}) = -\frac{b_1(\eta)}{c^4}\int \frac{2\mathsf{f}^{\alpha\beta}\Lambda_{\alpha\beta}(\eta', \mathbf{x}')d^3x'}{b_1(\eta')[3+2\omega(\eta')]|\mathbf{x}-\mathbf{x}'|}, \tag{24}$$

$$Q_{\mu\nu}(\eta, \mathbf{x}) = -4\frac{b_2(\eta)}{c^4}\int \frac{a^2(\eta')\Lambda_{\mu\nu}(\eta', \mathbf{x}')d^3x'}{b_2(\eta')\bar\phi(\eta')|\mathbf{x}-\mathbf{x}'|}, \tag{25}$$

with $b_1(\eta)$ and $b_2(\eta)$ satisfying the conditions

$$\frac{1}{b_1}\frac{db_1}{d\eta} = -\mathcal{H} + \frac{\mathcal{F}}{2} - \frac{\omega'\bar\phi\mathcal{F}}{3+2\omega}, \tag{26}$$

$$\frac{1}{b_2}\frac{db_2}{d\eta} = \mathcal{H} - \frac{\mathcal{F}}{2}. \tag{27}$$

Performing the near zone expansion [16] of (24) and (25) gives

$$\varphi = -\frac{2\mathsf{f}^{\alpha\beta}}{a(3+2\omega)c^4}\left[\Phi_{\alpha\beta} + \frac{\mathcal{F}}{2c}\left(1 + \frac{2\omega'\bar\phi}{3+2\omega}\right)\Psi_{\alpha\beta}\right], \tag{28}$$

$$Q_{\mu\nu} = -\frac{4a}{\bar\phi c^4}\left(\Phi_{\mu\nu} + \frac{\mathcal{F}}{2c}\Psi_{\alpha\beta}\right), \tag{29}$$

where

$$\Phi_{\mu\nu} = \int \frac{a\Lambda_{\mu\nu}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \frac{1}{c}\frac{d}{d\eta}\int a\Lambda_{\mu\nu}d^3x + \frac{1}{2c^2}\frac{d^2}{d\eta^2}\int a\Lambda_{\mu\nu}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x', \tag{30}$$

$$\Psi_{\mu\nu} = \int a\Lambda_{\mu\nu}d^3x - \frac{1}{c}\frac{d}{d\eta}\int a\Lambda_{\mu\nu}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|d^3x', \tag{31}$$

from which the metric perturbation is found to be

$$\varkappa_{\mu\nu} = \frac{4a}{\bar\phi c^4}\left\{\Phi_{\mu\nu} - \frac{\mathsf{f}_{\mu\nu}}{2}\left(1 - \frac{1}{3+2\omega}\right)\mathsf{f}^{\alpha\beta}\Phi_{\alpha\beta} + \frac{\mathcal{F}}{2c}\left[\Psi_{\mu\nu} - \frac{\mathsf{f}_{\mu\nu}}{2}\left(1 - \frac{1}{3+2\omega} - \frac{2\omega'\bar\phi}{(3+2\omega)^2}\right)\mathsf{f}^{\alpha\beta}\Psi_{\alpha\beta}\right]\right\}. \tag{32}$$

To simplify (32), we appeal to the covariant continuity of the stress-energy tensor $\Lambda_{\mu\nu}$, which in zeroth order in perturbations can be expressed in the form

$$\partial_0(a\Lambda_{00}) - \partial_j(a\Lambda_{0j}) = -(\mathcal{H}/c)(a\Lambda_{kk}), \tag{33}$$

$$\partial_0(a\Lambda_{i0}) - \partial_j(a\Lambda_{ij}) = -(\mathcal{H}/c)(a\Lambda_{i0}), \tag{34}$$

with $\Lambda_{kk} = \delta^{ij}\Lambda_{ij}$, or, upon volume integration,

$$\int \partial_0(a\Lambda_{00})d^3x = -\frac{\mathcal{H}}{c}\int (a\Lambda_{kk})d^3x, \tag{35}$$

$$\int \partial_0(a\Lambda_{i0})d^3x = -\frac{\mathcal{H}}{c}\int (a\Lambda_{i0})d^3x. \tag{36}$$

We notice that for any three-vector $\psi_i(\eta,\mathbf{x})$, $\psi_i = \partial_j(\psi_j x^i) - (\partial_j\psi_j)x^i$, and, upon integrating and discarding the divergence terms, $\int \psi_i d^3x = \int(-\partial_j\psi_j)x^i d^3x$. Applying this to $\psi_i \equiv a\Lambda_{i0}$ gives the virial relation,

$$\int (a\Lambda_{i0})d^3x = -\frac{1}{c}\frac{d}{d\eta}\int (a\Lambda_{00})x^i d^3x - \frac{\mathcal{H}}{c}\int (a\Lambda_{kk})x^i d^3x. \tag{37}$$

If we now define the mass, M, of the localized system by

$$M \equiv (1/c^2)\int (a\Lambda_{00})d^3x, \tag{38}$$

as well as its momentum, $P^i \equiv -(1/c)\int (a\Lambda_{i0})d^3x$, and dipole moment, $I^i \equiv (1/c^2)\int (a\Lambda_{00})x^i d^3x$, and restrict consideration to slowly moving sources only (that is, ignore all terms containing Λ_{ij}), we find that in the linearized approximation the system's mass is conserved,

$$dM/d\eta = 0, \tag{39}$$

and that momentum and dipole moment are related to each other by $P^i = dI^i/d\eta$. This allows us to introduce the system's rest frame in which the system's momentum vanishes, $P^i = 0$, provided the origin of coordinates is chosen at the system's center of mass, $I^i = 0$. Combining this with the monopole approximation, $|\mathbf{x}-\mathbf{x}'| \approx |\mathbf{x}|$, leads to $\Phi_{00} = M/|\mathbf{x}|$, $\Phi_{0i} = 0$, $\Phi_{ij} = 0$, $\Psi_{00} = M$, $\Psi_{0i} = 0$, $\Psi_{ij} = 0$, and we get

$$\varkappa_{00} = \frac{2aM}{\bar{\phi}c^2}\left\{\left(1+\frac{1}{3+2\omega}\right)\frac{1}{|\mathbf{x}|} + \frac{\mathcal{F}}{2c}\left(1+\frac{1}{3+2\omega}+\frac{2\omega'\bar{\phi}}{(3+2\omega)^2}\right)\right\}, \tag{40}$$

$$\varkappa_{i0} = 0, \tag{41}$$

$$\varkappa_{ij} = \frac{2aM}{\bar{\phi}c^2}\left\{\left(1-\frac{1}{3+2\omega}\right)\frac{1}{|\mathbf{x}|} + \frac{\mathcal{F}}{2c}\left(1-\frac{1}{3+2\omega}-\frac{2\omega'\bar{\phi}}{(3+2\omega)^2}\right)\right\}\delta_{ij}. \tag{42}$$

Introducing the post-Newtonian (PPN) parameters [17],

$$\gamma = \frac{1+\omega}{2+\omega}, \qquad G = \left(\frac{4+2\omega}{3+2\omega}\right)\frac{1}{\bar{\phi}}, \tag{43}$$

$$\beta = 1 + \frac{\omega'\bar{\phi}}{(3+2\omega)(4+2\omega)^2}, \tag{44}$$

with G having the meaning of the experimentally observable gravitational "constant," brings $\varkappa_{\mu\nu}$ to a compact form,

$$\varkappa_{00} = \frac{2aM}{c^2}\left(\frac{G}{|\mathbf{x}|} - \frac{1}{2c}\frac{dG}{d\eta}\right), \tag{45}$$

$$\varkappa_{i0} = 0, \tag{46}$$

$$\varkappa_{ij} = \frac{2aM}{c^2}\left(\frac{\gamma G}{|\mathbf{x}|} - \frac{1}{2c}\frac{d(\gamma G)}{d\eta}\right)\delta_{ij}. \tag{47}$$

In terms of the cosmological time, $t_{\rm H}$, the full metric is thus given by

$$g_{00} = -1 + \frac{2M}{c^2}\left(\frac{G}{a|\mathbf{x}|} - \frac{1}{2c}\frac{dG}{dt_{\rm H}}\right), \tag{48}$$

$$g_{i0} = 0, \tag{49}$$

$$g_{ij} = \left[1 + \frac{2M}{c^2}\left(\frac{\gamma G}{a|\mathbf{x}|} - \frac{1}{2c}\frac{d(\gamma G)}{dt_{\rm H}}\right)\right]a^2\delta_{ij}, \tag{50}$$

which in the limit $\mathcal{H}, \mathcal{F} \to 0$ reproduces the standard BD result [5].

III. EQUATIONS OF MOTION FOR GRAVITATIONAL PROBES

To uncover the observational consequences of the found metric, we have to derive the equations of motion for point

probes. For that, we introduce the local inertial coordinates, (ct, X^i), associated with a freely falling Hubble observer,

$$ct = ct_{\rm H} + a^2 H \delta_{ij} x^i x^j/(2c), \quad X^i = a x^i, \quad H \equiv \dot{a}/a, \tag{51}$$

where the overdot represents differentiation with respect to t, and H stands for the usual Hubble constant. Denoting $r \equiv |\boldsymbol{X}|$, we find

$$g_{00} = -1 + \frac{2GM}{c^2 r} - \frac{M}{c^3}\frac{dG}{dt}, \tag{52}$$

$$g_{i0} = -\frac{2(1+\gamma)GMHX^i}{c^3 r}, \tag{53}$$

$$g_{ij} = \left[1 + \frac{2\gamma GM}{c^2 r} - \frac{M}{c^3}\frac{d(\gamma G)}{dt}\right]\delta_{ij}, \tag{54}$$

with the linearized post-Newtonian connection coefficients being

$$\Gamma^i_{00} = \frac{1}{c^2}\frac{GMX^i}{r^3}, \qquad \Gamma^i_{j0} = \frac{1}{c^3}\frac{d(\gamma G)}{dt}\frac{M}{r}\delta_{ij},$$
$$\Gamma^i_{jk} = -\frac{1}{c^2}\frac{\gamma GM(\delta_{ij}X^k + \delta_{ik}X^j - \delta_{jk}X^i)}{r^3},$$
$$\Gamma^0_{00} = -\frac{1}{c^3}\frac{dG}{dt}\frac{M}{r}, \quad \Gamma^0_{j0} = \frac{1}{c^2}\frac{MGX^j}{r^3}, \quad \Gamma^0_{jk} = 0. \tag{55}$$

These are substituted into the geodesic equation parametrized by the coordinate time,

$$\frac{d^2X^i}{dt^2} = -c^2\Gamma^i_{00} - 2c\Gamma^i_{j0}\frac{dX^j}{dt} - \Gamma^i_{jk}\frac{dX^j}{dt}\frac{dX^k}{dt} + \left(c\Gamma^0_{00} + 2\Gamma^0_{j0}\frac{dX^j}{dt} + \frac{1}{c}\Gamma^0_{jk}\frac{dX^j}{dt}\frac{dX^k}{dt}\right)\frac{dX^i}{dt}, \tag{56}$$

with the result (here, $\boldsymbol{n} \equiv \boldsymbol{r}/r$, $\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$, $v \equiv |\boldsymbol{v}|$)

$$\ddot{\boldsymbol{r}} = -G(t)M\boldsymbol{n}/r^2 + \boldsymbol{F}, \tag{57}$$

where the disturbing force per unit mass is given by

$$\boldsymbol{F} = -\frac{\gamma GM}{c^2}\frac{v^2}{r^2}\boldsymbol{n} + \frac{(2\gamma+2\beta)G^2M^2}{c^2 r^3}\boldsymbol{n} + \frac{GM}{c^2}\left\{(2+2\gamma)\frac{\dot{r}}{r^2} - \left[(1+2\gamma)\frac{\dot{G}}{G} + 2\dot{\gamma}\right]\frac{1}{r}\right\}\boldsymbol{v}. \tag{58}$$

On the right-hand side of Eq. (58) we have included the "standard" first-order post-Newtonian (1PN) quadratic term (even though it does not formally follow from our linearized theory), which is expected on physical grounds.

We are particularly interested in the effect of Eq. (58) on Keplerian orbits. It is immediately clear that the following result of standard general relativity, with $\dot{G}/G = 0$, $\dot{\gamma} = 0$, holds: in the FLRW universe, in the linear Hubble approximation, planetary orbits do not change. The scalar-tensor theory, however, modifies that conclusion, as will be demonstrated below.

Because $d[\boldsymbol{r}\times\boldsymbol{v}]/dt \propto [\boldsymbol{r}\times\boldsymbol{v}]$, the motion is confined to a fixed orbital plane. This allows us to simplify the description of post-Newtonian dynamics by taking the orbital plane to coincide with the (X, Y) plane of the coordinate system [16]. Introducing the orbital basis [16,18], $\boldsymbol{n} = [\cos f, \sin f, 0]$, $\boldsymbol{\lambda} = [-\sin f, \cos f, 0]$, $\boldsymbol{e}_z = [0,0,1]$, in which $\boldsymbol{v} = \dot{r}\boldsymbol{n} + r\dot{f}\boldsymbol{\lambda}$, where f is the true anomaly (the orbital angle measured relative to the pericenter), brings $\boldsymbol{F}$ to the form

$$\boldsymbol{F} = \mathcal{R}\boldsymbol{n} + \mathcal{S}\boldsymbol{\lambda}, \tag{59}$$

where

$$\mathcal{R} = -\frac{GM}{c^2}\left\{\gamma\frac{v^2}{r^2} - (2+2\gamma)\frac{\dot{r}^2}{r^2} - (2\gamma+2\beta)\frac{GM}{r^3} + \left[(1+2\gamma)\frac{\dot{G}}{G} + 2\dot{\gamma}\right]\frac{\dot{r}}{r}\right\}, \tag{60}$$

$$\mathcal{S} = +\frac{GM}{c^2}\left\{(2+2\gamma)\frac{\dot{r}\dot{f}}{r} - \left[(1+2\gamma)\frac{\dot{G}}{G} + 2\dot{\gamma}\right]\dot{f}\right\}. \tag{61}$$

IV. SECULAR EVOLUTION

To find the secular changes of the orbital elements, a (semimajor axis, not to be confused with the cosmological scale factor), e (eccentricity), and ϖ (longitude of pericenter), we use the osculating equations of the perturbed Gyldén-Meshcherskii problem [19,20] (also see [21]),

$$\frac{da}{df} = \left\{-a\left[1 + \frac{2e}{1-e^2}(e+\cos f)\right]\frac{\dot{G}}{G} + \frac{2}{n\sqrt{1-e^2}}[e\sin f\mathcal{R} + (1+e\cos f)\mathcal{S}]\right\}\frac{dt}{df}, \tag{62}$$

$$\frac{de}{df} = \left\{-(e+\cos f)\frac{\dot{G}}{G} + \frac{\sqrt{1-e^2}}{na}\left[\sin f\mathcal{R} + \left(\cos f + \frac{e+\cos f}{1+e\cos f}\right)\mathcal{S}\right]\right\}\frac{dt}{df}, \tag{63}$$

$$\frac{d\varpi}{df} = \left\{-\frac{\sin f}{e}\frac{\dot{G}}{G} + \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\mathcal{S}\right]\right\}\frac{dt}{df}, \tag{64}$$

$$\frac{dt}{df}=\left\{\frac{n(1+e\cos f)^2}{(1-e^2)^{3/2}}+\frac{\sin f}{e}\frac{\dot G}{G}-\frac{\sqrt{1-e^2}}{nae}\left[-\cos f\mathcal{R}+\frac{2+e\cos f}{1+e\cos f}\sin f\mathcal{S}\right]\right\}^{-1}, \tag{65}$$

where

$$n(t)\equiv\sqrt{G(t)M/a^3(t)}=2\pi/P(t) \tag{66}$$

is the osculating mean motion, with $P(t)$ being the osculating orbital period. We write

$$G(t)=G_0+\dot G_0 t,\qquad \dot G_0/G_0\equiv s_1H, \tag{67}$$

$$\gamma(t)=\gamma_0+\dot\gamma_0 t,\qquad \dot\gamma_0\equiv s_2H, \tag{68}$$

$$\beta(t)=\beta_0+\dot\beta_0 t,\qquad \dot\beta_0\equiv s_3H, \tag{69}$$

where s_1, s_2, and s_3 are the adjustable parameters to be fixed by observations. Substituting the usual Keplerian relations for r, $\dot r$, $\dot f$, v^2 in $\mathcal{R}$ and $\mathcal{S}$, and using in Eqs. (62), (63), (64), (65) the zeroth-order orbital elements [16,22] and the values of G, γ, and β taken at the initial epoch, we get, upon integrating each of Eqs. (62), (63), and (64) with respect to f from 0 to 2π, the following 1PN changes per anomalistic period:

$$\langle\dot a\rangle=-aHs_1+\frac{H}{c^2}\frac{n^2a^3}{e^2\sqrt{1-e^2}}\left\{\left[2\left(-3+e^2+3\sqrt{1-e^2}\right)\beta_0+4\left(-1+\sqrt{1-e^2}\right)(6+5\gamma_0)\right.\right.$$
$$\left.\left.+e^2\left(4+2\sqrt{1-e^2}-4\gamma_0+5\sqrt{1-e^2}\gamma_0\right)\right]s_1+4e^2\left(-2+\sqrt{1-e^2}\right)s_2\right\}, \tag{70}$$

$$\langle\dot e\rangle=\frac{H}{c^2}\frac{n^2a^2\sqrt{1-e^2}}{e^3}\left\{\left[2\left(-2+e^2+2\sqrt{1-e^2}\right)\beta_0+2\left(-1+\sqrt{1-e^2}\right)(8+7\gamma_0)\right.\right.$$
$$\left.\left.+e^2\left(6+2\sqrt{1-e^2}+2\gamma_0+5\sqrt{1-e^2}\gamma_0\right)\right]s_1+4e^2\left(-1+\sqrt{1-e^2}\right)s_2\right\}, \tag{71}$$

$$\langle\dot\varpi\rangle=\frac{n^3a^2(2+2\gamma_0-\beta_0)}{c^2(1-e^2)}+\mathcal{O}(H^2). \tag{72}$$

Notice that in the limit $e\to1$ our approximation breaks down, as $\langle\dot a\rangle$ increases without bound. For $e\to0$, Eq. (71) gives $\langle\dot e\rangle\to0$, as had to be expected. The leading contribution in (72) is the standard 1PN general relativistic correction; in the linearized Hubble approximation, there is no additional contribution to the advance of the pericenter coming from the scalar-tensor theory.

To find the anomalistic rate of change of the osculating period we write (66) in the variational form,

$$\delta P=2\pi\delta\left(\sqrt{\frac{a^3}{GM}}\right)=P\left(\frac{3}{2}\frac{\delta a}{a}-\frac{1}{2}\frac{\delta G}{G}\right). \tag{73}$$

Taking δP to represent the anomalistic change of the osculating period and denoting $\delta P/P\equiv\langle\dot P\rangle$, we get upon substituting (67) and (70) in (73)

$$\frac{\langle\dot P\rangle}{P}=-2\frac{\dot G}{G}+\frac{H}{c^2}\frac{3n^2a^2}{2e^2\sqrt{1-e^2}}\{[2(-3+e^2+3\sqrt{1-e^2})\beta_0$$
$$+4(-1+\sqrt{1-e^2})(6+5\gamma_0)+e^2(4+2\sqrt{1-e^2}$$
$$-4\gamma_0+5\sqrt{1-e^2}\gamma_0)]s_1+4e^2(-2+\sqrt{1-e^2})s_2\}. \tag{74}$$

The first term on the right-hand side of (74) reproduces the result of [23]; the term proportional to H/c^2 extends it to the post-Newtonian domain.

Next, with the definition of β given in (43), we see that

$$\frac{\dot G}{G}=-\left(1-\frac{4(\beta-1)}{\gamma-1}\right)\mathcal{F}, \tag{75}$$

$$\dot\gamma=-\frac{4(\beta-1)(1+\gamma)}{\gamma-1}\mathcal{F}, \tag{76}$$

and thus s_1 and s_2 are related to each other via

$$s_2=\frac{4(\beta-1)(1+\gamma)}{\gamma-1-4(\beta-1)}s_1. \tag{77}$$

The constraints [12,24,25],

$$\gamma-1=2.3\times10^{-5},\qquad \beta-1=8\times10^{-5}, \tag{78}$$

$$\dot G_0/G_0=1.4\times10^{-13}\ \mathrm{yr}^{-1}, \tag{79}$$

$$H=7\times10^{-11}\ \mathrm{yr}^{-1}, \tag{80}$$

then give the estimates for s_1 and s_2,

$$s_1 \simeq 0.002, \qquad s_2 \simeq -0.004, \tag{81}$$

which, in turn, result in the following estimated secular changes per century for, say, the Hulse-Taylor binary:

$$(\Delta a)_{0\mathrm{PN}} \simeq -0.027 \text{ m}, \tag{82}$$

$$(\Delta a)_{1\mathrm{PN}} \simeq -7.3 \times 10^{-7} \text{ m}, \tag{83}$$

$$(\Delta e)_{1\mathrm{PN}} \simeq -6.3 \times 10^{-17}. \tag{84}$$

These are too small to be detectable with presently available technology.

V. SUMMARY

In conclusion, we performed post-Newtonian analysis of the equations of motion in the scalar-tensor theory of gravity for localized astronomical systems subjected to the time-dependent cosmological background. Several new cosmologically driven correction terms have been identified, and their effects on the secular evolution of the orbital elements have been calculated. At the present level of observational astronomy, these contributions are negligible and cannot affect any realistic analysis of orbital motion based on Eq. (2) [23]. However, should experimental methods develop further, the found corrections may prove helpful in establishing much stricter observational bounds on various PPN parameters as well as on the variability of the universal gravitational constant.

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Post-Newtonian celestial mechanics in scalar-tensor cosmology (Supplemental materials: Derivation of the wave equations for perturbations)

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Detailed derivation of the wave equations for cosmological perturbations of the scalar-tensor theory used in [Phys. Rev. D **94**, 044015 (2016)] is provided.

Contents

I. NOTATION

- T and $X^i = \{X, Y, Z\}$ are the coordinate time and isotropic spatial coordinates on the background manifold (in various parts of the manuscript other conventions may be used; e. g., in subsections of Section VIII);
- $X^\alpha = \{X^0, X^i\} = \{c\eta, X^i\}$ are the conformal coordinates with η being the conformal time;
- $x^\alpha = \{x^0, x^i\} = \{ct, x^i\}$ is an arbitrary coordinate chart on the background manifold;
- Greek indices $\alpha, \beta\gamma, \ldots, \mu, \nu, \ldots$ run through values 0, 1, 2, 3, and label spacetime coordinates;
- Roman indices $i, j, k, \ldots$ take values 1, 2, 3, and label spatial coordinates;
- Einstein summation convention for repeated (dummy) indices is always assumed, for example, $P^\alpha Q_\alpha \equiv P^0 Q_0 + P^1 Q_1 + P^2 Q_2 + P^3 Q_3$, and $P^i Q_i \equiv P^1 Q_1 + P^2 Q_2 + P^3 Q_3$;
- $g_{\alpha\beta}$ is a full metric on the cosmological spacetime manifold;
- $\bar{g}_{\alpha\beta}$ is the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric on the background spacetime manifold;
- $\mathfrak{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ is the (Gothic raised) metric tensor density of weight +1;
- $\bar{\mathfrak{g}}^{\alpha\beta} = \sqrt{-\bar{g}} \bar{g}^{\alpha\beta}$ is the background metric tensor density of weight +1;
- $f_{\alpha\beta}$ is the metric on the conformal spacetime manifold;
- $\eta_{\alpha\beta} = \mathrm{diag}\{-1, +1, +1, +1\}$ is the Minkowski metric;
- $R = R(T)$, or $a = a(\eta) = R[T(\eta)]$ is the scale factor of the FLRW metric;
- $H = R^{-1} dR/dT$ is the Hubble parameter;
- $\mathcal{H} = a^{-1} da/d\eta$ is the conformal Hubble parameter;
- a bar over a geometric object (as in $\bar{F}$), denotes the unperturbed value of F on the background manifold;
- the tensor indices of geometric objects on the background manifold are raised and lowered with the background metric $\bar{g}_{\alpha\beta}$, for example $F_{\alpha\beta} = \bar{g}_{\alpha\mu} \bar{g}_{\beta\nu} F^{\mu\nu}$;
- the tensor indices of geometric objects on the conformal spacetime are raised and lowered with the conformal metric $f_{\alpha\beta}$;
- symmetrization of a geometric object with respect to two indices is denoted with the parenthesis, $F_{(\alpha\beta)} \equiv (1/2)(F_{\alpha\beta} + F_{\beta\alpha})$;
- antisymmetrization of a geometric object with respect to two indices is denoted with the square brackets, $F_{[\alpha\beta]} \equiv (1/2)(F_{\alpha\beta} - F_{\beta\alpha})$;
- a prime, $W' = dW/d\phi$, denotes the derivative with respect to the scalar field ϕ;
- a dot, $\dot{F} = dF/d\eta$, denotes the total derivative with respect to the conformal time η;
- $\partial_\alpha = \partial/\partial x^\alpha$ is a partial derivative with respect to coordinate x^α;
- a comma followed by an index, $F_{,\alpha} \equiv \partial_\alpha F$, indicates the partial derivative with respect to coordinate x^α, which is a convenient notation in some cases. When no confusion may arise, the comma as a symbol of the partial derivative is omitted. For example, we may denote the partial derivatives of the scalar field by $\varphi_\alpha \equiv \varphi_{,\alpha}$;
- a vertical bar, $F_{|\alpha}$, denotes the covariant derivative associated with the background metric $\bar{g}_{\alpha\beta}$. Covariant derivatives of scalar fields coincide with their partial derivatives;
- a semicolon, $F_{;\alpha}$ denotes the covariant derivative associated with the conformal metric $f_{\alpha\beta}$
- ∇_α denotes the covariant derivative associated with the full metric $g_{\alpha\beta}$;
- ϕ is the fundamental scalar field of the Brans-Dicke theory;

- ω is the Brans-Dicke parameter; in general, $\omega = \omega(\phi)$;
- $\bar{\phi}$ is the background value of the Brans-Dicke (scalar) field ϕ;
- $\varphi = \phi - \bar{\phi}$ is the perturbation of ϕ from its background value $\bar{\phi}$. Fields ϕ and $\bar{\phi}$ refer to the same point on the spacetime manifold;
- $\varkappa_{\alpha\beta} \equiv g_{\alpha\beta} - \bar{g}_{\alpha\beta}$ is the metric tensor perturbation. Fields $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ refer to the same point on the spacetime manifold;
- $\mathfrak{h}_{\alpha\beta} \equiv \mathfrak{g}_{\alpha\beta} - \bar{\mathfrak{g}}_{\alpha\beta}$ is the perturbation of the metric density;
- $l^{\alpha\beta} \equiv \mathfrak{h}_{\alpha\beta}/\sqrt{-\bar{g}}$. In the linear approximation, $l^{\alpha\beta} = \varkappa^{\alpha\beta} + (1/2)\bar{g}^{\alpha\beta}\varkappa^{\alpha}{}_{\alpha}$, where $\varkappa^{\alpha}{}_{\alpha} = \bar{g}^{\alpha\beta}\varkappa_{\alpha\beta}$;
- the Christoffel symbols, $\Gamma^{\alpha}{}_{\beta\gamma} = (1/2)g^{\alpha\kappa}(g_{\kappa\gamma,\beta} + g_{\kappa\beta,\gamma} - g_{\beta\gamma,\kappa})$;
- the Riemann tensor, $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\mu\kappa}\Gamma^{\kappa}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\nu\kappa}\Gamma^{\kappa}{}_{\beta\mu}$;
- the Ricci tensor, $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$;
- the Ricci scalar, $R = g^{\alpha\beta}R_{\alpha\beta}$.

II. BRIEF REVIEW OF DYNAMICAL PERTURBATION THEORY

In accordance with the dynamical perturbation theory of spacetime manifolds developed in Refs. [1, 2] we write the variables of the theory as the sums of their background values and the corresponding perturbations,

$$\Phi^{A} = \bar{\Phi}^{A} + \varphi^{A}, \tag{1}$$

with Φ^{A} representing the generic multi-component field whose components are labeled by a generic index A. For example, Φ^{A} may collectively represent the metric density,

$$\mathfrak{g}^{\alpha\beta} = \bar{\mathfrak{g}}^{\alpha\beta} + \mathfrak{h}^{\alpha\beta}, \tag{2}$$

and the scalar field,

$$\phi = \bar{\phi} + \varphi, \tag{3}$$

of the Brans-Dicke theory. Denoting by $\mathcal{L}$ the Lagrangian of the theory (regarded as a function of Φ^{A} and its derivatives of arbitrary, but finite, order), we first notice that the variational derivative of $\mathcal{L}$ obeys the rule

$$\frac{\delta\mathcal{L}}{\delta\varphi^{A}} = \frac{\delta\mathcal{L}}{\delta\bar{\Phi}^{A}}, \tag{4}$$

which will be used in what follows. Expanding $\mathcal{L}$ in a Taylor series around $\bar{\Phi}^{A}$ gives

$$\mathcal{L} = \bar{\mathcal{L}} + \mathcal{L}_{1} + \mathcal{L}^{\mathrm{dyn}} + \mathcal{L}^{\mathrm{p}}, \tag{5}$$

where $\bar{\mathcal{L}} \equiv \mathcal{L}(\bar{\Phi}^{A})$ is the background Lagrangian,

$$\mathcal{L}_{1} \equiv \varphi^{A}\frac{\delta\bar{\mathcal{L}}}{\delta\bar{\Phi}^{A}}, \tag{6}$$

$\mathcal{L}^{\mathrm{dyn}}$ is the infinite sum of the higher-order terms in φ^{A} (in the linearized approximation, these are systematically discarded), and $\mathcal{L}^{\mathrm{p}}$ is the Lagrangian of a localized gravitational source (such as, e. g., a star or a planet), which is considered as a bare perturbation of the dynamical system). Because the barred variables satisfy the background field equations, we have

$$\frac{\delta\bar{\mathcal{L}}}{\delta\bar{\Phi}^{A}} = 0, \tag{7}$$

which constitutes the so-called *on-shell* condition. The dynamical perturbation theory is then based on the assumption that the evolution of the field perturbations is governed by the variational equation (now $\mathcal{L}$ is formally regarded as the function of φ^A),

$$\frac{\delta\mathcal{L}}{\delta\varphi^A}=0, \tag{8}$$

subject to (7). Thus, applying (4), (5) and (7) to (8) gives

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\varphi^A} &= \frac{\delta}{\delta\bar{\Phi}^A}\left(\bar{\mathcal{L}}+\mathcal{L}_1+\mathcal{L}^{\rm dyn}+\mathcal{L}^{\rm p}\right)\\ &= \frac{\delta}{\delta\bar{\Phi}^A}\left(\mathcal{L}_1+\mathcal{L}^{\rm dyn}+\mathcal{L}^{\rm p}\right)=0,\end{aligned} \tag{9}$$

which results in the field equations for perturbations,

$$\frac{2\kappa}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\Phi}^A}\left(\varphi^B\frac{\delta\bar{\mathcal{L}}}{\delta\bar{\Phi}^B}+\mathcal{L}^{\rm dyn}+\mathcal{L}^{\rm p}\right)=0, \tag{10}$$

where the prefactor was inserted for future convenience, with $\kappa\equiv 8\pi$ being (dimensionless) Einstein's gravitational constant. Notice that in (10) the on-shell condition (7) should not be imposed until after all the variational derivatives have been calculated.

III. DERIVATION OF THE WAVE EQUATIONS FOR PERTURBATIONS IN THE SCALAR-TENSOR THEORY

Our main goal is to derive the wave equations for scalar field and metric perturbations of the scalar-tensor theory, (77) and (78), which reproduce Eqs. (15) and (16) of Ref. [5].

A. Lagrangian and stress-energy tensor

We work with the Lagrangian of the form

$$\mathcal{L}^G = -\frac{1}{16\pi}\sqrt{-g}R\phi, \tag{11}$$

$$\mathcal{L}^{BD} = \sqrt{-g}\left[\frac{1}{2}\tilde{\omega}(\phi)g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}+W(\phi)\right], \tag{12}$$

where

$$\tilde{\omega}(\phi)=\frac{2}{16\pi}\frac{\omega}{\phi},\quad \tilde{\omega}'(\phi)=\frac{2}{16\pi}\frac{\omega}{\phi}\left(\frac{\omega'}{\omega}-\frac{1}{\phi}\right),\quad W(\phi)=\frac{2}{16\pi}\lambda\phi,\quad W'(\phi)=\frac{2}{16\pi}(\lambda'\phi+\lambda). \tag{13}$$

Notice that the associated stress-energy tensor of the scalar field is given by

$$T^{BD}_{\alpha\beta} = \tilde{\omega}(\phi)\phi_{,\alpha}\phi_{,\beta}-g_{\alpha\beta}\left[\frac{1}{2}\tilde{\omega}(\phi)g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}+W(\phi)\right]. \tag{14}$$

B. Background equations

Upon direct variational calculation, we find the following background field equations (here written in terms of $\tilde{\omega}$ and $\bar{W}$),

$$\bar{R}_{\mu\nu}\bar{\phi} = 8\pi\left(\bar{T}^M_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{T}^M+\bar{g}_{\mu\nu}\bar{W}+\tilde{\omega}\bar{\phi}_{|\mu}\bar{\phi}_{|\nu}\right)+\bar{\phi}_{|\mu\nu}+\frac{1}{2}\bar{g}_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha}, \tag{15}$$

$$\bar{R}\bar{\phi} = 8\pi\left(-\bar{T}^M+4\bar{W}+\tilde{\omega}\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\right)+3\bar{\phi}^{|\alpha}_{|\alpha}, \tag{16}$$

$$\bar{\phi}^{|\alpha}_{|\alpha} = \frac{8\pi}{3+16\pi\tilde{\omega}\bar{\phi}}\left[\bar{T}^M-\left(\tilde{\omega}+\tilde{\omega}'\bar{\phi}\right)\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}-4\bar{W}+2\bar{W}'\bar{\phi}\right]. \tag{17}$$

Alternatively, in terms of $\omega \equiv \bar{\omega}$ and $\bar{W}$,

$$\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right)\bar{\phi} = 8\pi\left(\bar{T}^M_{\mu\nu} - \bar{g}_{\mu\nu}\bar{W}\right) + \frac{\omega}{\bar{\phi}}\left(\bar{\phi}_{|\mu}\bar{\phi}_{|\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\right) + \bar{\phi}_{|\mu\nu} - \bar{g}_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha}, \tag{18}$$

$$\bar{\phi}^{|\alpha}_{|\alpha} = \frac{1}{3+2\omega}\left(8\pi\bar{T}^M - \omega'\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha} - 4\bar{W} + 2\bar{W}'\bar{\phi}\right), \tag{19}$$

which immediately shows that

$$\bar{T}^M_{\mu\nu} \sim \bar{W} \sim \mathcal{O}(\mathcal{H}^2). \tag{20}$$

Also, in terms of ω and λ,

$$\bar{R}_{\mu\nu} = \frac{8\pi}{\bar{\phi}}\left(\bar{T}^M_{\mu\nu} - \frac{1+\omega}{3+2\omega}\bar{g}_{\mu\nu}\bar{T}^M\right) + \frac{\bar{g}_{\mu\nu}}{3+2\omega}\left[2(1+\omega)\lambda + \lambda'\bar{\phi} - \frac{1}{2}\frac{\omega'}{\bar{\phi}}\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\right] + \frac{1}{\bar{\phi}}\bar{\phi}_{|\mu\nu} + \frac{\omega}{\bar{\phi}^2}\bar{\phi}_{|\mu}\bar{\phi}_{|\nu}, \tag{21}$$

$$\bar{R} = -\frac{8\pi}{\bar{\phi}}\frac{2\omega}{3+2\omega}\bar{T}^M + \frac{2(3+4\omega)\lambda}{3+2\omega} + \frac{6\lambda'\bar{\phi}}{3+2\omega} + \left(\frac{\omega}{\bar{\phi}} - \frac{3\omega'}{3+2\omega}\right)\frac{\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}}{\bar{\phi}}, \tag{22}$$

$$\bar{\phi}^{|\alpha}_{|\alpha} = \frac{1}{3+2\omega}\left(8\pi\bar{T}^M - 2\lambda\bar{\phi} + 2\lambda'\bar{\phi}^2 - \omega'\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\right). \tag{23}$$

In the above,

$$\bar{\phi}_{|\mu\nu} = \bar{\phi}_{,\mu\nu} - \bar{\Gamma}^\rho_{\mu\nu}\bar{\phi}_{,\rho}, \tag{24}$$

$$\bar{\phi}^{|\alpha}_{|\alpha} = \bar{g}^{\alpha\beta}\left(\bar{\phi}_{,\alpha\beta} - \bar{\Gamma}^\rho_{\alpha\beta}\bar{\phi}_{,\rho}\right), \tag{25}$$

$$\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha} = \bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}. \tag{26}$$

C. Equation for $l_{\mu\nu}$ perturbation

In accordance with the dynamical perturbation theory of spacetime manifolds developed in Ref. [1], the field equations for metric perturbations, Eqs. (10), are

$$F^G_{\mu\nu} + F^{BD}_{\mu\nu} = 8\pi\Lambda_{\mu\nu}, \tag{27}$$

where

$$F^G_{\mu\nu} = \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\phi}}\right) \tag{28}$$

$$F^{BD}_{\mu\nu} = \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\phi}}\right), \tag{29}$$

and $\Lambda_{\mu\nu}$ is the stress-energy tensor of the localized source. Notice that in deriving (27) we defined the stress-energy tensor of the source via

$$\Lambda_{\mu\nu} \equiv +\frac{2}{\sqrt{-\bar{g}}}\frac{\delta\mathcal{L}^{\rm P}}{\delta\bar{g}^{\mu\nu}}, \tag{30}$$

treated $\mathcal{L}^{\rm P}$ as being of first order of smallness, and used the chain rule,

$$\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} = \frac{\delta\bar{g}^{\rho\sigma}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\frac{\delta}{\delta\bar{g}^{\rho\sigma}} = \frac{1}{2\sqrt{-\bar{g}}}(\delta^\rho_\alpha\delta^\sigma_\beta + \delta^\rho_\beta\delta^\sigma_\alpha - \bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma})\frac{\delta}{\delta\bar{g}^{\rho\sigma}}. \tag{31}$$

1. Derivation of $F^G_{\mu\nu}$

We have,

$$\begin{aligned}
-16\pi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} &= -16\pi\frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{g}^{\mu\nu}} \\
&= -16\pi\frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\frac{\partial\bar{g}_{\alpha\beta}}{\partial\bar{g}^{\mu\nu}}\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{g}_{\alpha\beta}} \\
&= -16\pi\frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\left(-\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\right)\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{g}_{\alpha\beta}} \\
&= \frac{1}{2\sqrt{-\bar{g}}}\left(\delta^\mu_\rho\delta^\nu_\sigma+\delta^\mu_\sigma\delta^\nu_\rho-\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right)\left(-\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\right)\left(-16\pi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{g}_{\alpha\beta}}\right). \qquad (32)
\end{aligned}$$

Now, using (99),

$$\begin{aligned}
-16\pi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{g}_{\mu\nu}} &= \frac{\delta}{\delta\bar{g}_{\mu\nu}}\left(\sqrt{-\bar{g}}\bar{\phi}\bar{R}\right) \\
&= \frac{\delta}{\delta\bar{g}_{\mu\nu}}\left(\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\bar{R}^\rho_{\lambda\gamma\kappa}\right) \\
&= \frac{\partial\left(\sqrt{-\bar{g}}\bar{g}^{\lambda\kappa}\right)}{\partial\bar{g}_{\mu\nu}}\bar{\phi}\delta^\gamma_\rho\bar{R}^\rho_{\lambda\gamma\kappa}+\left[\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\left(\bar{g}^{\sigma\nu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\alpha}}+\bar{g}^{\sigma\mu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\alpha\beta\nu}}-\bar{g}^{\sigma\alpha}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\nu}}\right)\right]_{|\beta\alpha} \\
&= \left[\frac{\partial\left(\sqrt{-\bar{g}}\right)}{\partial\bar{g}_{\mu\nu}}\bar{g}^{\lambda\kappa}+\sqrt{-\bar{g}}\frac{\partial\bar{g}^{\lambda\kappa}}{\partial\bar{g}_{\mu\nu}}\right]\bar{\phi}\delta^\gamma_\rho\bar{R}^\rho_{\lambda\gamma\kappa}+\left[\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\left(\bar{g}^{\sigma\nu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\alpha}}+\bar{g}^{\sigma\mu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\alpha\beta\nu}}-\bar{g}^{\sigma\alpha}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\nu}}\right)\right]_{|\beta\alpha} \\
&= \left[\left(+\frac{1}{2}\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\right)\bar{g}^{\lambda\kappa}+\sqrt{-\bar{g}}\left(-\bar{g}^{\lambda\mu}\bar{g}^{\kappa\nu}\right)\right]\bar{\phi}\bar{R}_{\lambda\kappa}+\left[\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\left(\bar{g}^{\sigma\nu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\alpha}}+\bar{g}^{\sigma\mu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\alpha\beta\nu}}-\bar{g}^{\sigma\alpha}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\nu}}\right)\right]_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi}+\left[\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\left(\bar{g}^{\sigma\nu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\alpha}}+\bar{g}^{\sigma\mu}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\alpha\beta\nu}}-\bar{g}^{\sigma\alpha}\frac{\partial\bar{R}^\rho_{\lambda\gamma\kappa}}{\partial\bar{R}^\sigma_{\mu\beta\nu}}\right)\right]_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi} \\
&\quad+\left[\sqrt{-\bar{g}}\bar{\phi}\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\left(\bar{g}^{\sigma\nu}\delta^\rho_\sigma\delta^\mu_\lambda\frac{\delta^\beta_\gamma\delta^\alpha_\kappa-\delta^\beta_\kappa\delta^\alpha_\gamma}{2}+\bar{g}^{\sigma\mu}\delta^\rho_\sigma\delta^\alpha_\lambda\frac{\delta^\beta_\gamma\delta^\nu_\kappa-\delta^\beta_\kappa\delta^\nu_\gamma}{2}-\bar{g}^{\sigma\alpha}\delta^\rho_\sigma\delta^\mu_\lambda\frac{\delta^\beta_\gamma\delta^\nu_\kappa-\delta^\beta_\kappa\delta^\nu_\gamma}{2}\right)\right]_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi} \\
&\quad+\frac{\sqrt{-\bar{g}}}{2}\left[\bar{\phi}\left(\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\bar{g}^{\sigma\nu}\delta^\rho_\sigma\delta^\mu_\lambda\left(\delta^\beta_\gamma\delta^\alpha_\kappa-\delta^\beta_\kappa\delta^\alpha_\gamma\right)+\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\bar{g}^{\sigma\mu}\delta^\rho_\sigma\delta^\alpha_\lambda\left(\delta^\beta_\gamma\delta^\nu_\kappa-\delta^\beta_\kappa\delta^\nu_\gamma\right)-\bar{g}^{\lambda\kappa}\delta^\gamma_\rho\bar{g}^{\sigma\alpha}\delta^\rho_\sigma\delta^\mu_\lambda\left(\delta^\beta_\gamma\delta^\nu_\kappa-\delta^\beta_\kappa\delta^\nu_\gamma\right)\right)\right]_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi}+\frac{\sqrt{-\bar{g}}}{2}\left[\bar{\phi}\left(\left(\bar{g}^{\mu\alpha}\bar{g}^{\beta\nu}-\bar{g}^{\mu\beta}\bar{g}^{\alpha\nu}\right)+\left(\bar{g}^{\alpha\nu}\bar{g}^{\beta\mu}-\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\right)-\left(\bar{g}^{\mu\nu}\bar{g}^{\beta\alpha}-\bar{g}^{\mu\beta}\bar{g}^{\nu\alpha}\right)\right)\right]_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi}+\frac{\sqrt{-\bar{g}}}{2}\left(\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}+\bar{g}^{\alpha\nu}\bar{g}^{\beta\mu}-2\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\right)\bar{\phi}_{|\beta\alpha} \\
&= \sqrt{-\bar{g}}\left[\left(\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}-\bar{R}^{\mu\nu}\right)\bar{\phi}+\frac{\bar{\phi}^{|\mu\nu}+\bar{\phi}^{|\nu\mu}}{2}-\bar{g}^{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha}\right] \\
&= -\sqrt{-\bar{g}}\left[\left(\bar{R}^{\mu\nu}-\frac{1}{2}\bar{g}^{\mu\nu}\bar{R}\right)\bar{\phi}+\bar{g}^{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha}-\bar{\phi}^{|\mu\nu}\right]. \qquad (33)
\end{aligned}$$

Substituting (33) in (32) gives

$$-16\pi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} = \frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma+\delta^\mu_\sigma\delta^\nu_\rho-\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right)\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\left[\left(\bar{R}^{\alpha\beta}-\frac{1}{2}\bar{g}^{\alpha\beta}\bar{R}\right)\bar{\phi}+\bar{g}^{\alpha\beta}\bar{\phi}^{|\kappa}_{|\kappa}-\bar{\phi}^{|\alpha\beta}\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right)\left[\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right)\bar{\phi} + \bar{g}_{\mu\nu}\bar{\phi}^{|\kappa}_{|\kappa} - \bar{\phi}_{|\mu\nu}\right] \\
&= \left(\bar{R}_{\rho\sigma} - \frac{1}{2}\bar{g}_{\rho\sigma}\bar{R}\right)\bar{\phi} + \bar{g}_{\rho\sigma}\bar{\phi}^{|\kappa}_{|\kappa} - \bar{\phi}_{|\rho\sigma} - \frac{1}{2}\bar{g}_{\rho\sigma}\left[\left(\bar{R} - 2\bar{R}\right)\bar{\phi} + 4\bar{\phi}^{|\kappa}_{|\kappa} - \bar{g}^{\mu\nu}\bar{\phi}_{|\mu\nu}\right] \\
&= \bar{R}_{\rho\sigma}\bar{\phi} - \frac{1}{2}\bar{g}_{\rho\sigma}\bar{\phi}^{|\kappa}_{|\kappa} - \bar{\phi}_{|\rho\sigma}. \qquad (34)
\end{aligned}
$$

Additionally,

$$
-16\pi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\phi}} = \sqrt{-\bar{g}}\bar{R}. \qquad (35)
$$

Thus, from (34) and (35),

$$
-16\pi\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\phi}}\right) = \mathfrak{h}^{\rho\sigma}\left(\bar{R}_{\rho\sigma}\bar{\phi} - \frac{1}{2}\bar{g}_{\rho\sigma}\bar{\phi}^{|\kappa}_{|\kappa} - \bar{\phi}_{|\rho\sigma}\right) + \varphi\sqrt{-\bar{g}}\bar{R}. \qquad (36)
$$

To get (28) we still have to take the variational derivative of (36) with respect to $\bar{g}^{\mu\nu}$.

First, by analogy with (33), and taking into account an extra minus sign due to the derivative being with respect to the *raised* metric, we have

$$
\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\varphi\sqrt{-\bar{g}}\bar{R}\right) = \left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right)\varphi + \bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha} - \varphi_{|\mu\nu}. \qquad (37)
$$

Next, by analogy with general relativity,

$$
\begin{aligned}
\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\bar{\phi}\right) &= \frac{1}{2}\left[\left(\bar{\phi}l_{\mu\nu}\right)^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}\left(\bar{\phi}l^{\alpha\beta}\right)_{|\alpha\beta} - \left(\bar{\phi}l^\alpha{}_\mu\right)_{|\nu\alpha} - \left(\bar{\phi}l^\alpha{}_\nu\right)_{|\mu\alpha}\right] \\
&= \frac{1}{2}\left(\bar{\phi}^{|\alpha}_{|\alpha}l_{\mu\nu} + 2\bar{\phi}^{|\alpha}l_{\mu\nu|\alpha} + \bar{\phi}l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}\right) \\
&\quad + \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta} + 2\bar{\phi}_{|\alpha}l^{\alpha\beta}{}_{|\beta} + \bar{\phi}l^{\alpha\beta}{}_{|\alpha\beta}\right) \\
&\quad - \frac{1}{2}\left(\bar{\phi}_{|\nu\alpha}l^\alpha{}_\mu + \bar{\phi}_{|\nu}l^\alpha{}_{\mu|\alpha} + \bar{\phi}_{|\alpha}l^\alpha{}_{\mu|\nu} + \bar{\phi}l^\alpha{}_{\mu|\nu\alpha}\right) \\
&\quad - \frac{1}{2}\left(\bar{\phi}_{|\mu\alpha}l^\alpha{}_\nu + \bar{\phi}_{|\mu}l^\alpha{}_{\nu|\alpha} + \bar{\phi}_{|\alpha}l^\alpha{}_{\nu|\mu} + \bar{\phi}l^\alpha{}_{\nu|\mu\alpha}\right). \qquad (38)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\frac{1}{2}\mathfrak{h}^{\rho\sigma}\bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}\bar{\phi}_{|\alpha\beta}\right) = -\frac{\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\kappa\lambda}}\left[\mathfrak{h}^{\rho\sigma}\bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}\left(\bar{\phi}_{,\alpha\beta} - \bar{\Gamma}^\gamma_{\alpha\beta}\bar{\phi}_{,\gamma}\right)\right] \\
&= \frac{-\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2\sqrt{-\bar{g}}}\left\{\mathfrak{h}^{\kappa\lambda}\bar{g}^{\alpha\beta}\bar{\phi}_{|\alpha\beta} - \mathfrak{h}^{\rho\sigma}\bar{g}_{\rho\sigma}\bar{g}^{\alpha\kappa}\bar{g}^{\beta\lambda}\bar{\phi}_{|\alpha\beta} - \frac{1}{2}\left[\left(g^{\chi\kappa}\frac{\partial}{\partial\bar{\Gamma}^\chi_{\lambda\pi}} + g^{\chi\lambda}\frac{\partial}{\partial\bar{\Gamma}^\chi_{\kappa\pi}} - g^{\chi\pi}\frac{\partial}{\partial\bar{\Gamma}^\chi_{\kappa\lambda}}\right)\mathfrak{h}^{\rho\sigma}\bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}\left(\bar{\phi}_{,\alpha\beta} - \bar{\Gamma}^\gamma_{\alpha\beta}\bar{\phi}_{,\gamma}\right)\right]_{|\pi}\right\} \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}\bar{g}^{\alpha\beta}\left[\left(\bar{g}^{\chi\kappa}\frac{\partial\bar{\Gamma}^\gamma_{\alpha\beta}}{\partial\bar{\Gamma}^\chi_{\lambda\pi}} + \bar{g}^{\chi\lambda}\frac{\partial\bar{\Gamma}^\gamma_{\alpha\beta}}{\partial\bar{\Gamma}^\chi_{\kappa\pi}} - \bar{g}^{\chi\pi}\frac{\partial\bar{\Gamma}^\gamma_{\alpha\beta}}{\partial\bar{\Gamma}^\chi_{\kappa\lambda}}\right)l\bar{\phi}_{,\gamma}\right]_{|\pi} \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}\bar{g}^{\alpha\beta}\left[\left(\bar{g}^{\chi\kappa}\delta^\gamma_\chi\frac{\delta^\lambda_\alpha\delta^\pi_\beta + \delta^\lambda_\beta\delta^\pi_\alpha}{2} + \bar{g}^{\chi\lambda}\delta^\gamma_\chi\frac{\delta^\kappa_\alpha\delta^\pi_\beta + \delta^\kappa_\beta\delta^\pi_\alpha}{2} - \bar{g}^{\chi\pi}\delta^\gamma_\chi\frac{\delta^\kappa_\alpha\delta^\lambda_\beta + \delta^\kappa_\beta\delta^\lambda_\alpha}{2}\right)l\bar{\phi}_{,\gamma}\right]_{|\pi} \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}\bar{g}^{\alpha\beta}\left[\left(\bar{g}^{\gamma\kappa}\frac{\delta^\lambda_\alpha\delta^\pi_\beta + \delta^\lambda_\beta\delta^\pi_\alpha}{2} + \bar{g}^{\gamma\lambda}\frac{\delta^\kappa_\alpha\delta^\pi_\beta + \delta^\kappa_\beta\delta^\pi_\alpha}{2} - \bar{g}^{\gamma\pi}\frac{\delta^\kappa_\alpha\delta^\lambda_\beta + \delta^\kappa_\beta\delta^\lambda_\alpha}{2}\right)l\bar{\phi}_{|\gamma}\right]_{|\pi} \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\left(\delta^\gamma_\mu\delta^\pi_\nu + \delta^\pi_\mu\delta^\gamma_\nu - \bar{g}_{\mu\nu}\bar{g}^{\gamma\pi}\right)\left(l\bar{\phi}_{|\gamma}\right)_{|\pi} \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\left[\left(l\bar{\phi}_{|\mu}\right)_{|\nu} + \left(l\bar{\phi}_{|\nu}\right)_{|\mu} - \bar{g}_{\mu\nu}\left(l\bar{\phi}_{|\alpha}\right)^{|\alpha}\right] \\
&= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\left[l_{|\nu}\bar{\phi}_{|\mu} + l\bar{\phi}_{|\mu\nu} + l_{|\mu}\bar{\phi}_{|\nu} + l\bar{\phi}_{|\nu\mu} - \bar{g}_{\mu\nu}l^{|\alpha}\bar{\phi}_{|\alpha} - \bar{g}_{\mu\nu}l\bar{\phi}^{|\alpha}_{|\alpha}\right]
\end{aligned}
$$

$$= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}l\bar{\phi}_{|\mu\nu} - \frac{1}{4}\left[l_{|\nu}\bar{\phi}_{|\mu} + l_{|\mu}\bar{\phi}_{|\nu} + 2l\bar{\phi}_{|\nu\mu} - \bar{g}_{\mu\nu}l^{|\alpha}\bar{\phi}_{|\alpha} - \bar{g}_{\mu\nu}l\bar{\phi}^{|\alpha}_{|\alpha}\right]$$
$$= -\frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} - \frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu} + l_{|\nu}\bar{\phi}_{|\mu}\right) + \frac{1}{4}\bar{g}_{\mu\nu}\left(l^{|\alpha}\bar{\phi}_{|\alpha} + l\bar{\phi}^{|\alpha}_{|\alpha}\right). \tag{39}$$

Also,

$$\begin{aligned}
\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\bar{\phi}_{|\rho\sigma}\right) &= -\frac{\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\kappa\lambda}}\left[\mathfrak{h}^{\rho\sigma}\left(\bar{\phi}_{,\rho\sigma} - \bar{\Gamma}^{\gamma}_{\rho\sigma}\bar{\phi}_{,\gamma}\right)\right] \\
&= \frac{-\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2\sqrt{-\bar{g}}}\left[\left(g^{\chi\kappa}\frac{\partial}{\partial\bar{\Gamma}^{\chi}_{\lambda\pi}} + g^{\chi\lambda}\frac{\partial}{\partial\bar{\Gamma}^{\chi}_{\kappa\pi}} - g^{\chi\pi}\frac{\partial}{\partial\bar{\Gamma}^{\chi}_{\kappa\lambda}}\right)\mathfrak{h}^{\rho\sigma}\bar{\Gamma}^{\gamma}_{\rho\sigma}\bar{\phi}_{,\gamma}\right]_{|\pi} \\
&= \frac{-\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2}\left[\left(\bar{g}^{\chi\kappa}\delta^{\gamma}_{\chi}\frac{\delta^{\lambda}_{\rho}\delta^{\pi}_{\sigma} + \delta^{\lambda}_{\sigma}\delta^{\pi}_{\rho}}{2} + \bar{g}^{\chi\lambda}\delta^{\gamma}_{\chi}\frac{\delta^{\kappa}_{\rho}\delta^{\pi}_{\sigma} + \delta^{\kappa}_{\sigma}\delta^{\pi}_{\rho}}{2} - \bar{g}^{\chi\pi}\delta^{\gamma}_{\chi}\frac{\delta^{\kappa}_{\rho}\delta^{\lambda}_{\sigma} + \delta^{\kappa}_{\sigma}\delta^{\lambda}_{\rho}}{2}\right)l^{\rho\sigma}\bar{\phi}_{|\gamma}\right]_{|\pi} \\
&= \frac{-\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2}\left[\left(\bar{g}^{\gamma\kappa}\frac{\delta^{\lambda}_{\rho}\delta^{\pi}_{\sigma} + \delta^{\lambda}_{\sigma}\delta^{\pi}_{\rho}}{2} + \bar{g}^{\gamma\lambda}\frac{\delta^{\kappa}_{\rho}\delta^{\pi}_{\sigma} + \delta^{\kappa}_{\sigma}\delta^{\pi}_{\rho}}{2} - \bar{g}^{\gamma\pi}\frac{\delta^{\kappa}_{\rho}\delta^{\lambda}_{\sigma} + \delta^{\kappa}_{\sigma}\delta^{\lambda}_{\rho}}{2}\right)l^{\rho\sigma}\bar{\phi}_{|\gamma}\right]_{|\pi} \\
&= \frac{-\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda}}{2}\left[\left(\bar{g}^{\gamma\kappa}l^{\lambda\pi} + \bar{g}^{\gamma\lambda}l^{\kappa\pi} - \bar{g}^{\gamma\pi}l^{\kappa\lambda}\right)\bar{\phi}_{|\gamma}\right]_{|\pi} \\
&= -\frac{1}{2}\left(l_{\nu}{}^{\pi}\bar{\phi}_{|\mu} + l_{\mu}{}^{\pi}\bar{\phi}_{|\nu} - l_{\mu\nu}\bar{\phi}^{|\pi}\right)_{|\pi} \\
&= -\frac{1}{2}\left[\left(\bar{\phi}_{|\mu}l_{\nu}{}^{\alpha}\right)_{|\alpha} + \left(\bar{\phi}_{|\nu}l_{\mu}{}^{\alpha}\right)_{|\alpha} - \left(\bar{\phi}^{|\alpha}l_{\mu\nu}\right)_{|\alpha}\right] \\
&= -\frac{1}{2}\left[\bar{\phi}_{|\mu\alpha}l_{\nu}{}^{\alpha} + \bar{\phi}_{|\nu\alpha}l_{\mu}{}^{\alpha} - \bar{\phi}^{|\alpha}_{|\alpha}l_{\mu\nu} + \bar{\phi}_{|\mu}l_{\nu}{}^{\alpha}{}_{|\alpha} + \bar{\phi}_{|\nu}l_{\mu}{}^{\alpha}{}_{|\alpha} - \bar{\phi}^{|\alpha}l_{\mu\nu|\alpha}\right].
\end{aligned} \tag{40}$$

Combining (38), (39), and (40) gives

$$\begin{aligned}
\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left[\mathfrak{h}^{\rho\sigma}\left(\bar{R}_{\rho\sigma}\bar{\phi} - \frac{1}{2}\bar{g}_{\rho\sigma}\bar{\phi}^{|\kappa}_{|\kappa} - \bar{\phi}_{|\rho\sigma}\right)\right] &= \frac{1}{2}\left(\underbrace{\bar{\phi}^{|\alpha}_{|\alpha}l_{\mu\nu}}_{5} + \underbrace{2\bar{\phi}^{|\alpha}l_{\mu\nu|\alpha}}_{[6]} + \bar{\phi}l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}\right) \\
&\quad + \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta} + 2\bar{\phi}_{|\alpha}l^{\alpha\beta}{}_{|\beta} + \bar{\phi}l^{\alpha\beta}{}_{|\alpha\beta}\right) \\
&\quad - \frac{1}{2}\left(\underbrace{\bar{\phi}_{|\nu\alpha}l^{\alpha}{}_{\mu}}_{2} + \underbrace{\bar{\phi}_{|\nu}l^{\alpha}{}_{\mu|\alpha}}_{4} + \bar{\phi}_{|\alpha}l^{\alpha}{}_{\mu|\nu} + \bar{\phi}l^{\alpha}{}_{\mu|\nu\alpha}\right) \\
&\quad - \frac{1}{2}\left(\underbrace{\bar{\phi}_{|\mu\alpha}l^{\alpha}{}_{\nu}}_{1} + \underbrace{\bar{\phi}_{|\mu}l^{\alpha}{}_{\nu|\alpha}}_{3} + \bar{\phi}_{|\alpha}l^{\alpha}{}_{\nu|\mu} + \bar{\phi}l^{\alpha}{}_{\nu|\mu\alpha}\right) \\
&\quad + \frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu} + l_{|\nu}\bar{\phi}_{|\mu}\right) - \frac{1}{4}\bar{g}_{\mu\nu}\left(l^{|\alpha}\bar{\phi}_{|\alpha} + l\bar{\phi}^{|\alpha}_{|\alpha}\right) \\
&\quad + \frac{1}{2}\left(\underbrace{\bar{\phi}_{|\mu\alpha}l_{\nu}{}^{\alpha}}_{1} + \underbrace{\bar{\phi}_{|\nu\alpha}l_{\mu}{}^{\alpha}}_{2} - \underbrace{\bar{\phi}^{|\alpha}_{|\alpha}l_{\mu\nu}}_{5} + \underbrace{\bar{\phi}_{|\mu}l_{\nu}{}^{\alpha}{}_{|\alpha}}_{3} + \underbrace{\bar{\phi}_{|\nu}l_{\mu}{}^{\alpha}{}_{|\alpha}}_{4} - \underbrace{\bar{\phi}^{|\alpha}l_{\mu\nu|\alpha}}_{[6]}\right) \\
&= \frac{1}{2}\left(\bar{\phi}^{|\alpha}l_{\mu\nu|\alpha} + \bar{\phi}l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}\right) \\
&\quad + \frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta} + 2\bar{\phi}_{|\alpha}l^{\alpha\beta}{}_{|\beta} + \bar{\phi}l^{\alpha\beta}{}_{|\alpha\beta}\right) \\
&\quad - \frac{1}{2}\left(\bar{\phi}_{|\alpha}l^{\alpha}{}_{\mu|\nu} + \bar{\phi}l^{\alpha}{}_{\mu|\nu\alpha}\right) - \frac{1}{2}\left(\bar{\phi}_{|\alpha}l^{\alpha}{}_{\nu|\mu} + \bar{\phi}l^{\alpha}{}_{\nu|\mu\alpha}\right) \\
&\quad + \frac{1}{2}l_{\mu\nu}\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu} + l_{|\nu}\bar{\phi}_{|\mu}\right) - \frac{1}{4}\bar{g}_{\mu\nu}\left(l^{|\alpha}\bar{\phi}_{|\alpha} + l\bar{\phi}^{|\alpha}_{|\alpha}\right) \\
&= \frac{1}{2}\bar{\phi}\left(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}l^{\alpha\beta}{}_{|\alpha\beta} - l^{\alpha}{}_{\mu|\nu\alpha} - l^{\alpha}{}_{\nu|\mu\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta}+2\bar{\phi}_{|\alpha}l^{\alpha\beta}{}_{|\beta}-\frac{1}{2}\bar{\phi}_{|\alpha}l^{|\alpha}\right)\\
&+\frac{1}{2}\bar{\phi}^{|\alpha}\left(l_{\mu\nu|\alpha}-l_{\alpha\mu|\nu}-l_{\alpha\nu|\mu}\right)\\
&+\frac{1}{2}\bar{\phi}^{|\alpha}_{|\alpha}\left(l_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}l\right)+\frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu}+l_{|\nu}\bar{\phi}_{|\mu}\right). \qquad (41)
\end{aligned}
$$

Combining (41) with (37) gives

$$
\begin{aligned}
F^G_{\mu\nu} \;=\; &\frac{1}{2}\bar{\phi}\underbrace{\left(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}+\bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha}-A_{\mu|\nu}-A_{\nu|\mu}-\bar{R}^{\alpha}_{\nu}l_{\mu\alpha}-\bar{R}^{\alpha}_{\mu}l_{\nu\alpha}-2\bar{R}_{\mu\alpha\beta\nu}l^{\alpha\beta}\right)}_{\equiv l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}+\bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha}-l^{\alpha}{}_{\mu|\nu\alpha}-l^{\alpha}{}_{\nu|\mu\alpha}}\\
&+\frac{1}{2}\bar{\phi}^{|\alpha}\left(l_{\mu\nu|\alpha}-l_{\alpha\mu|\nu}-l_{\alpha\nu|\mu}\right)+\frac{1}{2}\bar{\phi}^{|\alpha}_{|\alpha}\left(l_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}l\right)\\
&+\frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta}+2\bar{\phi}_{|\alpha}A^{\alpha}-\frac{1}{2}\bar{\phi}_{|\alpha}l^{|\alpha}\right)+\frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu}+l_{|\nu}\bar{\phi}_{|\mu}\right)\\
&+\left(\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right)\varphi+\bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha}-\varphi_{|\mu\nu}. \qquad (42)
\end{aligned}
$$

2. *Derivation of $F^{BD}_{\mu\nu}$*

We have,

$$
\begin{aligned}
\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} \;&=\; \frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{g}^{\mu\nu}}\\
&=\; \frac{1}{2\sqrt{-\bar{g}}}\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}+\delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}-\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right)\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{g}^{\mu\nu}}. \qquad (43)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{g}^{\mu\nu}} \;&=\; \frac{\delta}{\delta\bar{g}^{\mu\nu}}\left[\sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W(\bar{\phi})\right)\right]\\
&=\; -\frac{1}{2}\sqrt{-\bar{g}}\bar{g}_{\mu\nu}\left(\frac{1}{2}\tilde{\omega}(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W(\bar{\phi})\right)+\frac{1}{2}\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\mu}\bar{\phi}_{,\nu}\\
&=\; \frac{1}{2}\sqrt{-\bar{g}}\underbrace{\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\mu}\bar{\phi}_{,\nu}-\bar{g}_{\mu\nu}\left(\frac{1}{2}\tilde{\omega}(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W(\bar{\phi})\right)\right]}_{\equiv\bar{T}^{BD}_{\mu\nu}}. \qquad (44)
\end{aligned}
$$

Substituting (44) in (43) gives

$$
\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}=\frac{1}{2}\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\rho}\bar{\phi}_{,\sigma}+\bar{g}_{\rho\sigma}W(\bar{\phi})\right]. \qquad (45)
$$

Also, using (98),

$$
\begin{aligned}
\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\phi}} \;&=\; \frac{\partial\bar{\mathcal{L}}^{BD}}{\partial\bar{\phi}}-\left[\frac{\partial\bar{\mathcal{L}}^{BD}}{\partial\bar{\phi}_{|\rho}}\right]_{|\rho}\\
&=\; \frac{\partial}{\partial\bar{\phi}}\left[\sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W(\bar{\phi})\right)\right]-\left\{\frac{\partial}{\partial\bar{\phi}_{|\rho}}\left[\sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W(\bar{\phi})\right)\right]\right\}_{|\rho}\\
&=\; \sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}'(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W'(\bar{\phi})\right)-\left(\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}^{|\rho}\right)_{|\rho}. \qquad (46)
\end{aligned}
$$

Thus, from (45) and (46),

$$\begin{aligned}
\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}+\varphi\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\phi}} &= \mathfrak{h}^{\rho\sigma}\frac{1}{2}\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\rho}\bar{\phi}_{,\sigma}+\bar{g}_{\rho\sigma}W(\bar{\phi})\right]+\varphi\left[\sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}'(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W'(\bar{\phi})\right)-\left(\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}^{|\rho}\right)_{|\rho}\right]\\
&= \mathfrak{h}^{\rho\sigma}\frac{1}{2}\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\rho}\bar{\phi}_{,\sigma}+\bar{g}_{\rho\sigma}W(\bar{\phi})\right]\\
&\quad+\sqrt{-\bar{g}}\left(\frac{1}{2}\tilde{\omega}'(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W'(\bar{\phi})\right)\varphi+\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}^{|\rho}\varphi_{|\rho}-\underbrace{\left(\varphi\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}^{|\rho}\right)_{|\rho}}_{\text{cov. divergence}}\\
&= \mathfrak{h}^{\rho\sigma}\frac{1}{2}\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\rho}\bar{\phi}_{,\sigma}+\bar{g}_{\rho\sigma}W(\bar{\phi})\right]\\
&\quad+\sqrt{-\bar{g}}\left[\left(\frac{1}{2}\tilde{\omega}'(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta}+W'(\bar{\phi})\right)\varphi+\tilde{\omega}(\bar{\phi})\bar{g}^{\rho\sigma}\bar{\phi}_{,\sigma}\varphi_{,\rho}\right]-\underbrace{\left(\varphi\sqrt{-\bar{g}}\tilde{\omega}(\bar{\phi})\bar{\phi}^{|\rho}\right)_{|\rho}}_{\text{cov. divergence}}.
\end{aligned}\tag{47}$$

Taking the variational derivative of (47) with respect to $\bar{g}^{\mu\nu}$, dropping the covariant divergence term, and noticing that $\mathfrak{h}^{\rho\sigma}$ is independent of $\bar{g}^{\mu\nu}$ (even though we formally write $\mathfrak{h}^{\rho\sigma}\equiv\sqrt{-\bar{g}}l^{\rho\sigma}$), we get

$$\begin{aligned}
F^{BD}_{\mu\nu} &= \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}+\varphi\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\phi}}\right)\\
&= 8\pi\left[l^{\mu\nu}\bar{W}+\bar{g}_{\mu\nu}\left(\frac{1}{2}\tilde{\omega}'\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\varphi+\bar{W}'\varphi+\tilde{\omega}\bar{\phi}^{|\alpha}\varphi_{|\alpha}\right)-\tilde{\omega}'\bar{\phi}_{|\mu}\bar{\phi}_{|\nu}\varphi-\tilde{\omega}\left(\bar{\phi}_{|\mu}\varphi_{|\nu}+\bar{\phi}_{|\nu}\varphi_{|\mu}\right)\right].
\end{aligned}\tag{48}$$

3. *Final result*

Thus,

$$F^{G}_{\mu\nu}+F^{BD}_{\mu\nu}=8\pi\Lambda_{\mu\nu},\tag{49}$$

where

$$\begin{aligned}
F^{G}_{\mu\nu} &= \frac{1}{2}\bar{\phi}\underbrace{\left(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}+\bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha}-A_{\mu|\nu}-A_{\nu|\mu}-\bar{R}^{\alpha}_{\nu}l_{\mu\alpha}-\bar{R}^{\alpha}_{\mu}l_{\nu\alpha}-2\bar{R}_{\mu\alpha\beta\nu}l^{\alpha\beta}\right)}_{\equiv l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}+\bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha}-l^{\alpha}{}_{\mu|\nu\alpha}-l^{\alpha}{}_{\nu|\mu\alpha}}\\
&\quad+\frac{1}{2}\bar{\phi}^{|\alpha}\left(l_{\mu\nu|\alpha}-l_{\alpha\mu|\nu}-l_{\alpha\nu|\mu}\right)+\frac{1}{2}\bar{\phi}^{|\alpha}_{|\alpha}\left(l_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}l\right)\\
&\quad+\frac{1}{2}\bar{g}_{\mu\nu}\left(\bar{\phi}_{|\alpha\beta}l^{\alpha\beta}+2\bar{\phi}_{|\alpha}A^{\alpha}-\frac{1}{2}\bar{\phi}_{|\alpha}l^{|\alpha}\right)+\frac{1}{4}\left(l_{|\mu}\bar{\phi}_{|\nu}+l_{|\nu}\bar{\phi}_{|\mu}\right)\\
&\quad+\left(\bar{R}_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right)\varphi+\bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha}-\varphi_{|\mu\nu},
\end{aligned}\tag{50}$$

$$F^{BD}_{\mu\nu}=8\pi\left[l_{\mu\nu}\bar{W}+\bar{g}_{\mu\nu}\left(\frac{\tilde{\omega}'}{2}\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\varphi+\bar{W}'\varphi+\tilde{\omega}\bar{\phi}^{|\alpha}\varphi_{|\alpha}\right)-\tilde{\omega}'\bar{\phi}_{|\mu}\bar{\phi}_{|\nu}\varphi-\tilde{\omega}\left(\bar{\phi}_{|\mu}\varphi_{|\nu}+\bar{\phi}_{|\nu}\varphi_{|\mu}\right)\right],\tag{51}$$

$$A^{\alpha}\equiv l^{\alpha\beta}{}_{|\beta},\tag{52}$$

and

$$\tilde{\omega}=\tilde{\omega}(\bar{\phi}),\quad \tilde{\omega}'=\frac{d\tilde{\omega}(\bar{\phi})}{d\bar{\phi}},\quad \bar{W}=W(\bar{\phi}),\quad \bar{W}'=\frac{dW(\bar{\phi})}{d\bar{\phi}}.\tag{53}$$

D. Equation for l perturbation

Taking the trace of (49) gives

$$\bar{\phi}\left(\frac{1}{2}l^{|\alpha}{}_{|\alpha}+A^{\alpha}{}_{|\alpha}\right)-\frac{1}{2}\bar{\phi}^{|\alpha}_{|\alpha}l+2\bar{\phi}_{|\alpha\beta}l^{\alpha\beta}+3\bar{\phi}_{|\alpha}A^{\alpha}-\bar{R}\varphi+3\varphi^{|\alpha}{}_{|\alpha}+8\pi\left[\bar{W}l+\tilde{\omega}'\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\varphi+4\bar{W}'\varphi+2\tilde{\omega}\bar{\phi}^{|\alpha}\varphi_{|\alpha}\right]=8\pi\Lambda.\tag{54}$$

E. Equation for φ perturbation

1. Derivation

From general theory of Ref. [1],

$$F_A^G + F_A^{BD} = 0. \tag{55}$$

Using (10) and (98) we get

$$\begin{aligned} F_A^G &= \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\phi}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^G}{\delta\bar{\phi}}\right) \\ &= \frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\phi}}\left[\mathfrak{h}^{\rho\sigma}\left(\bar{R}_{\rho\sigma}\bar{\phi} - \frac{1}{2}\bar{g}_{\rho\sigma}\bar{g}^{\alpha\beta}\bar{\phi}_{|\alpha\beta} - \bar{\phi}_{|\rho\sigma}\right) + \varphi\sqrt{-\bar{g}}\bar{R}\right] \\ &= l^{\rho\sigma}\bar{R}_{\rho\sigma} - \frac{1}{2}\bar{g}^{\alpha\beta}l_{|\alpha\beta} - l^{\alpha\beta}{}_{|\alpha\beta}. \end{aligned} \tag{56}$$

and

$$\begin{aligned} F_A^{BD} &= \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\phi}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \varphi\frac{\delta\bar{\mathcal{L}}^{BD}}{\delta\bar{\phi}}\right) \\ &= \frac{-16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\phi}}\left\{\mathfrak{h}^{\rho\sigma}\frac{1}{2}\left[\tilde{\omega}(\bar{\phi})\bar{\phi}_{,\rho}\bar{\phi}_{,\sigma} + \bar{g}_{\rho\sigma}W(\bar{\phi})\right] + \sqrt{-\bar{g}}\left[\left(\frac{1}{2}\tilde{\omega}'(\bar{\phi})\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta} + W'(\bar{\phi})\right)\varphi + \tilde{\omega}(\bar{\phi})\bar{g}^{\rho\sigma}\bar{\phi}_{,\sigma}\varphi_{,\rho}\right]\right\} \\ &= -16\pi\left\{l^{\rho\sigma}\frac{1}{2}\left(\tilde{\omega}'\bar{\phi}_{|\rho}\bar{\phi}_{|\sigma} + \bar{g}_{\rho\sigma}\bar{W}'\right) + \left(\frac{1}{2}\tilde{\omega}''\bar{g}^{\alpha\beta}\bar{\phi}_{|\alpha}\bar{\phi}_{|\beta} + \bar{W}''\right)\varphi + \tilde{\omega}'\bar{g}^{\rho\sigma}\bar{\phi}_{|\sigma}\varphi_{|\rho}\right\} \\ &\quad +16\pi\left\{\left(l^{\rho\sigma}\tilde{\omega}\bar{\phi}_{|\rho}\right)_{|\sigma} + \left(\tilde{\omega}'\bar{g}^{\alpha\beta}\bar{\phi}_{|\beta}\varphi\right)_{|\alpha} + \left(\tilde{\omega}\varphi^{|\rho}\right)_{|\rho}\right\} \\ &= -16\pi\left\{l^{\rho\sigma}\frac{1}{2}\left(\tilde{\omega}'\bar{\phi}_{|\rho}\bar{\phi}_{|\sigma} + \bar{g}_{\rho\sigma}\bar{W}'\right) + \left(\frac{1}{2}\tilde{\omega}''\bar{g}^{\alpha\beta}\bar{\phi}_{|\alpha}\bar{\phi}_{|\beta} + \bar{W}''\right)\varphi + \tilde{\omega}'\bar{g}^{\rho\sigma}\bar{\phi}_{|\sigma}\varphi_{|\rho}\right\} \\ &\quad +16\pi\left\{\tilde{\omega}l^{\rho\sigma}{}_{|\sigma}\bar{\phi}_{|\rho} + \tilde{\omega}'l^{\rho\sigma}\bar{\phi}_{|\rho}\bar{\phi}_{|\sigma} + \tilde{\omega}l^{\rho\sigma}\bar{\phi}_{|\rho\sigma}\right\} \\ &\quad +16\pi\left\{\tilde{\omega}''\bar{g}^{\alpha\beta}\bar{\phi}_{|\beta}\bar{\phi}_{|\alpha}\varphi + \tilde{\omega}'\bar{g}^{\alpha\beta}\bar{\phi}_{|\beta\alpha}\varphi + \tilde{\omega}'\bar{g}^{\alpha\beta}\bar{\phi}_{|\beta}\varphi_{|\alpha}\right\} \\ &\quad +16\pi\left\{\tilde{\omega}'\bar{\phi}_{|\alpha}\varphi^{|\alpha} + \tilde{\omega}\varphi^{|\alpha}{}_{|\alpha}\right\} \\ &= 16\pi\left\{\tilde{\omega}\varphi^{|\alpha}{}_{|\alpha} + \tilde{\omega}'\bar{\phi}_{|\alpha}\varphi^{|\alpha} + \frac{1}{2}\tilde{\omega}''\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\varphi + \tilde{\omega}'\bar{\phi}^{|\alpha}_{|\alpha}\varphi - \bar{W}''\varphi + \frac{1}{2}\tilde{\omega}'l^{\rho\sigma}\bar{\phi}_{|\rho}\bar{\phi}_{|\sigma} + \tilde{\omega}l^{\rho\sigma}{}_{|\sigma}\bar{\phi}_{|\rho} + \tilde{\omega}l^{\rho\sigma}\bar{\phi}_{|\rho\sigma} - \frac{1}{2}\bar{W}'l\right\}. \end{aligned} \tag{57}$$

Combining (56) and (57) gives

$$\begin{aligned} &16\pi\left\{\tilde{\omega}\varphi^{|\alpha}{}_{|\alpha} + \tilde{\omega}'\bar{\phi}_{|\alpha}\varphi^{|\alpha} + \frac{1}{2}\tilde{\omega}''\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha}\varphi + \tilde{\omega}'\bar{\phi}^{|\alpha}_{|\alpha}\varphi - \bar{W}''\varphi + \frac{1}{2}\tilde{\omega}'l^{\rho\sigma}\bar{\phi}_{|\rho}\bar{\phi}_{|\sigma} + \tilde{\omega}l^{\rho\sigma}{}_{|\sigma}\bar{\phi}_{|\rho} + \tilde{\omega}l^{\rho\sigma}\bar{\phi}_{|\rho\sigma} - \frac{1}{2}\bar{W}'l\right\} \\ &+l^{\rho\sigma}\bar{R}_{\rho\sigma} - \frac{1}{2}\bar{g}^{\alpha\beta}l_{|\alpha\beta} - l^{\alpha\beta}{}_{|\alpha\beta} = 0. \end{aligned} \tag{58}$$

2. Final result

$$\begin{aligned} &\varphi^{|\alpha}{}_{|\alpha} + \frac{\tilde{\omega}'}{\tilde{\omega}}\bar{\phi}^{|\alpha}\varphi_{|\alpha} + \frac{1}{\tilde{\omega}}\left(\tilde{\omega}'\bar{\phi}^{|\alpha}_{|\alpha} + \frac{1}{2}\tilde{\omega}''\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha} - \bar{W}''\right)\varphi \\ &\quad +A^{\alpha}\bar{\phi}_{|\alpha} + l^{\alpha\beta}\bar{\phi}_{|\alpha\beta} + \frac{1}{2}\frac{\tilde{\omega}'}{\tilde{\omega}}l^{\alpha\beta}\bar{\phi}_{|\alpha}\bar{\phi}_{|\beta} - \frac{1}{2}\frac{\bar{W}'}{\tilde{\omega}}l + \frac{1}{16\pi}\frac{1}{\tilde{\omega}}\left(\bar{R}_{\alpha\beta}l^{\alpha\beta} - \frac{1}{2}l^{|\alpha}{}_{|\alpha} - A^{\alpha}{}_{|\alpha}\right) = 0. \end{aligned} \tag{59}$$

IV. LINEAR HUBBLE APPROXIMATION IN THE SCALAR-TENSOR THEORY

In the linear Hubble approximation for perturbations φ and $l_{\mu\nu}$ we ignore the terms containing $\mathcal{H}^2$, $\dot{\mathcal{H}}$, and $\ddot{\bar{\phi}}/\bar{\phi}$. (The overdot, we recall, represents the derivative with respect to the conformal time, $d/d\eta$.) To keep track of the time dependence of the gravitational "constant", despite of (191), we must retain all terms containing $\dot{\bar{\phi}}/\bar{\phi}$.

Additionally, we will use the following formula valid in isotropic conformal coordinates:

$$\varphi^{|\alpha}{}_{|\alpha} \;=\; \frac{1}{a^2}\left(\Box\varphi - 2\mathcal{H}\varphi_{,0}\right). \tag{60}$$

A. Field equations

To arrive at the linear Hubble approximation, we first drop the "obvious" terms proportional to $\bar{\phi}^{|\mu}_{|\nu}$, $\bar{\phi}_{|\mu}\bar{\phi}_{|\nu}$, $\bar{R}_{\mu\nu}$, $\bar{W}$, $\bar{W}'$, in Eqs. (54), (59), (49), and get the following three equations,

$$\bar{\phi}\left(\frac{1}{2}l^{|\alpha}{}_{|\alpha} + A^{\alpha}{}_{|\alpha}\right) + 3\bar{\phi}_{|\alpha}A^{\alpha} + 3\varphi^{|\alpha}{}_{|\alpha} + \frac{2\omega}{\bar{\phi}}\bar{\phi}^{|\alpha}\varphi_{|\alpha} = 8\pi\Lambda, \tag{61}$$

$$\frac{1}{2}l^{|\alpha}{}_{|\alpha} - \frac{2\omega}{\bar{\phi}}\varphi^{|\alpha}{}_{|\alpha} - \frac{2\omega}{\bar{\phi}}\left(\frac{\omega'}{\omega} - \frac{1}{\bar{\phi}}\right)\bar{\phi}^{|\alpha}\varphi_{|\alpha} - \frac{2\omega}{\bar{\phi}}A^{\alpha}\bar{\phi}_{|\alpha} + A^{\alpha}{}_{|\alpha} = 0, \tag{62}$$

and

$$\begin{aligned}
&\left(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha} - A_{\mu|\nu} - A_{\nu|\mu}\right) + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(l_{\mu\nu|\alpha} - l_{\alpha\mu|\nu} - l_{\alpha\nu|\mu}\right)\\
&-\bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(\frac{1}{2}l_{|\alpha} - \frac{2\omega}{\bar{\phi}}\varphi_{|\alpha}\right) + \frac{\bar{\phi}_{|\mu}}{\bar{\phi}}\left(\frac{1}{2}l_{|\nu} - \frac{2\omega}{\bar{\phi}}\varphi_{|\nu}\right) + \frac{\bar{\phi}_{|\nu}}{\bar{\phi}}\left(\frac{1}{2}l_{|\mu} - \frac{2\omega}{\bar{\phi}}\varphi_{|\mu}\right)\\
&+2\bar{g}_{\mu\nu}\frac{\bar{\phi}_{|\alpha}}{\bar{\phi}}A^{\alpha} + \frac{2}{\bar{\phi}}\left(\bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha} - \varphi_{|\mu\nu}\right) = \frac{16\pi}{\bar{\phi}}\Lambda_{\mu\nu},
\end{aligned} \tag{63}$$

where, we recall, $A^{\alpha} \equiv l^{\alpha\beta}{}_{|\beta}$, $\omega \equiv \omega(\bar{\phi})$, $\omega' \equiv d\omega/d\bar{\phi}$. Instead of (61), by combining (61) and (62), we get

$$\varphi^{|\alpha}{}_{|\alpha} + \frac{2\omega'}{3+2\omega}\bar{\phi}^{|\alpha}\varphi_{|\alpha} + A^{\alpha}\bar{\phi}_{|\alpha} = \frac{8\pi}{3+2\omega}\Lambda, \tag{64}$$

which will be used in what follows. Thus, we have the system of equations (64) and (63). Substituting

$$-\frac{2\omega}{\bar{\phi}}\varphi_{|\alpha} = \left(-\frac{2\omega}{\bar{\phi}}\varphi\right)_{|\alpha} + \mathcal{O}(\mathcal{H}) \tag{65}$$

in (63), we get to order $\mathcal{O}(\mathcal{H})$,

$$\begin{aligned}
&\left(l_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \underbrace{\bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha} - A_{\mu|\nu} - A_{\nu|\mu}}_{\text{gauge}}\right) + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(l_{\mu\nu|\alpha} \quad \underbrace{-l_{\alpha\mu|\nu} - l_{\alpha\nu|\mu}}_{\text{will cancel by } C^{\alpha} \text{ gauge}}\right)\\
&\underbrace{-\bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\alpha} + \frac{\bar{\phi}_{|\mu}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\nu} + \frac{\bar{\phi}_{|\nu}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\mu}}_{\text{will cancel by } C^{\alpha} \text{ gauge}}\\
&+2\bar{g}_{\mu\nu}\frac{\bar{\phi}_{|\alpha}}{\bar{\phi}}A^{\alpha} + \frac{2}{\bar{\phi}}\left(\bar{g}_{\mu\nu}\varphi^{|\alpha}{}_{|\alpha} - \varphi_{|\mu\nu}\right) = \frac{16\pi}{\bar{\phi}}\Lambda_{\mu\nu}.
\end{aligned} \tag{66}$$

We now introduce the gauge

$$A^{\alpha} = B^{\alpha} + C^{\alpha} + D^{\alpha}, \tag{67}$$

where we define

$$B^\alpha \equiv \underbrace{-\frac{2\mathcal{H}}{a} l^{\alpha\beta}\bar{u}_\beta}_{\text{"Celestial..."}}, \tag{68}$$

$$C^\alpha \equiv -\frac{\bar{\phi}^{|\beta}}{\bar{\phi}} l_\beta{}^\alpha + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right), \tag{69}$$

$$D^\alpha \equiv \underbrace{-\frac{\varphi^{|\alpha}}{\bar{\phi}}}_{\text{Brans-Dicke}} - \frac{2\mathcal{H}}{a}\bar{u}^\alpha\frac{\varphi}{\bar{\phi}} - \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}^2}\varphi. \tag{70}$$

This gauge generalizes both the gauge used in the "Celestial ephemerides" paper [3], and the gauge used in the original Brans-Dicke paper [4]. We notice that to order $O(\mathcal{H})$,

$$l_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}B^\alpha{}_{|\alpha} - B_{\mu|\nu} - B_{\nu|\mu} = \bar{g}^{\alpha\beta}l_{\mu\nu,\alpha\beta} + \frac{2\mathcal{H}}{a}\bar{u}^\alpha\partial_\alpha l_{\mu\nu} + \frac{2\mathcal{H}}{a}\left(\bar{g}_{\mu\nu}\bar{u}^\alpha\frac{\varphi_{|\alpha}}{\bar{\phi}} - \bar{u}_\mu\frac{\varphi_{|\nu}}{\bar{\phi}} - \bar{u}_\nu\frac{\varphi_{|\mu}}{\bar{\phi}}\right), \tag{71}$$

$$\begin{aligned}\bar{g}_{\mu\nu}C^\alpha{}_{|\alpha} - C_{\mu|\nu} - C_{\nu|\mu} &= \bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\alpha} - \frac{\bar{\phi}_{|\mu}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\nu} - \frac{\bar{\phi}_{|\nu}}{\bar{\phi}}\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)_{|\mu} \\ &\quad + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\left(l_{\alpha\mu|\nu} + l_{\alpha\nu|\mu}\right) + \bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}\varphi_{|\alpha}}{\bar{\phi}^2},\end{aligned} \tag{72}$$

$$\bar{g}_{\mu\nu}D^\alpha{}_{|\alpha} - D_{\mu|\nu} - D_{\nu|\mu} = -\bar{g}_{\mu\nu}\frac{\varphi^{|\alpha}{}_{|\alpha}}{\bar{\phi}} + \frac{2\varphi_{|\mu\nu}}{\bar{\phi}} - \frac{2\mathcal{H}}{a}\left(\bar{g}_{\mu\nu}\bar{u}^\alpha\frac{\varphi_{|\alpha}}{\bar{\phi}} - \bar{u}_\mu\frac{\varphi_{|\nu}}{\bar{\phi}} - \bar{u}_\nu\frac{\varphi_{|\mu}}{\bar{\phi}}\right), \tag{73}$$

where in (71) we used Eq. (115) of Sec. VI. This gives the system,

$$\begin{aligned}\varphi^{|\alpha}{}_{|\alpha} + \left(\frac{2\omega'\bar{\phi}}{3+2\omega} - 1\right)\frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\varphi_{|\alpha} &= \frac{8\pi}{3+2\omega}\Lambda, \\ \bar{g}^{\alpha\beta}l_{\mu\nu,\alpha\beta} + \left(\frac{2\mathcal{H}}{a}\bar{u}^\alpha + \frac{\bar{\phi}^{|\alpha}}{\bar{\phi}}\right)l_{\mu\nu,\alpha} + \bar{g}_{\mu\nu}\frac{\varphi^{|\alpha}{}_{|\alpha}}{\bar{\phi}} - \bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}\varphi_{|\alpha}}{\bar{\phi}^2} &= \frac{16\pi}{\bar{\phi}}\Lambda_{\mu\nu}.\end{aligned} \tag{74}$$

Re-writing everything in the Hubble conformal coordinates with the help of (60), and taking into account that, to order $O(\mathcal{H})$,

$$\Box\left(\frac{a^2\varphi}{\bar{\phi}}\right) = \frac{a^2}{\bar{\phi}}\Box\varphi - 2(2\mathcal{H} - \mathcal{F})\left(\frac{a^2\varphi}{\bar{\phi}}\right)_{,0}, \tag{75}$$

and

$$\begin{aligned}\bar{g}_{\mu\nu}\frac{\varphi^{|\alpha}{}_{|\alpha}}{\bar{\phi}} &= \frac{a^2 f_{\mu\nu}}{\bar{\phi}}\frac{1}{a^2}\left(\Box\varphi - 2\mathcal{H}\varphi_{,0}\right) \\ &= \frac{f_{\mu\nu}}{a^2}\left[\frac{a^2}{\bar{\phi}}\Box\varphi - 2\mathcal{H}\left(\frac{a^2\varphi}{\bar{\phi}}\right)_{,0}\right] \\ &= \frac{f_{\mu\nu}}{a^2}\left[\Box\left(\frac{a^2\varphi}{\bar{\phi}}\right) + 2(2\mathcal{H} - \mathcal{F})\left(\frac{a^2\varphi}{\bar{\phi}}\right)_{,0} - 2\mathcal{H}\left(\frac{a^2\varphi}{\bar{\phi}}\right)_{,0}\right] \\ &= \frac{1}{a^2}\left[\Box\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right) + (2\mathcal{H} - \mathcal{F})\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right)_{,0} - \mathcal{F}\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right)_{,0}\right] \\ &= \frac{1}{a^2}\left[\Box\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right) + (2\mathcal{H} - \mathcal{F})\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right)_{,0}\right] + a^2 f_{\mu\nu}\left(-\frac{1}{a^2}\right)\mathcal{F}\frac{\varphi_{,0}}{\bar{\phi}} \\ &= \frac{1}{a^2}\left[\Box\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right) + (2\mathcal{H} - \mathcal{F})\left(\frac{a^2 f_{\mu\nu}\varphi}{\bar{\phi}}\right)_{,0}\right] + \bar{g}_{\mu\nu}\frac{\bar{\phi}^{|\alpha}\varphi_{|\alpha}}{\bar{\phi}^2},\end{aligned} \tag{76}$$

we get

$$\Box\varphi - 2\left(\mathcal{H} - \frac{\mathcal{F}}{2} + \frac{\omega'\bar{\phi}}{3+2\omega}\mathcal{F}\right)\varphi_{,0} = \frac{8\pi}{3+2\omega}f^{\alpha\beta}\Lambda_{\alpha\beta}, \tag{77}$$

$$\Box Q_{\mu\nu} + 2\left(\mathcal{H} - \frac{\mathcal{F}}{2}\right)Q_{\mu\nu,0} = \frac{16\pi a^2}{\bar{\phi}}\Lambda_{\mu\nu}, \tag{78}$$

which reproduces Eqs. (15) and (16) of Ref. [5]. In the above, we defined

$$\mathcal{F} \equiv \frac{\bar{\phi}_{,0}}{\bar{\phi}} = \frac{1}{\bar{\phi}}\frac{d\bar{\phi}}{d\eta}, \tag{79}$$

and introduced a new gravitational variable $Q_{\mu\nu}$ (a direct analogue of the variable α_{ij} that appears in Eq. (23) of the original Brans-Dicke paper [4]),

$$Q_{\mu\nu} \equiv l_{\mu\nu} + \bar{g}_{\mu\nu}\frac{\varphi}{\bar{\phi}}. \tag{80}$$

Additionally, and this will turn out to be important for checking the gauge condition in Section VII, substituting

$$-\frac{2\omega}{\bar{\phi}}\varphi^{|\alpha}{}_{|\alpha} = \left(-\frac{2\omega}{\bar{\phi}}\varphi\right)^{|\alpha}_{\ \ |\alpha} + \mathcal{O}(\mathcal{H}), \tag{81}$$

in (62), we get to order $\mathcal{O}(\mathcal{H})$,

$$\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)^{|\alpha}_{\ \ |\alpha} + A^{\alpha}{}_{|\alpha} + \mathcal{O}(\mathcal{H}) = 0. \tag{82}$$

Using (67) gives

$$A^{\alpha}{}_{|\alpha} = -\frac{\varphi^{|\alpha}{}_{|\alpha}}{\bar{\phi}} + \mathcal{O}(\mathcal{H}), \tag{83}$$

and thus, from (82),

$$\left(\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi\right)^{|\alpha}_{\ \ |\alpha} = \frac{\varphi^{|\alpha}{}_{|\alpha}}{\bar{\phi}} + \mathcal{O}(\mathcal{H}), \tag{84}$$

which shows that the field perturbations satisfy the constraint

$$\frac{1}{2}l - \frac{2\omega}{\bar{\phi}}\varphi = \frac{\varphi}{\bar{\phi}} + \mathcal{O}(\mathcal{H}), \tag{85}$$

and thus the actual form of the gauge satisfied by the field perturbations is not (67), but a somewhat simpler,

$$A^{\alpha} = -\frac{2\mathcal{H}}{a}l^{\alpha\beta}\bar{u}_{\beta} - \frac{\bar{\phi}^{|\beta}}{\bar{\phi}}l_{\beta}{}^{\alpha} - \frac{2\mathcal{H}}{a}\bar{u}^{\alpha}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|\alpha}}{\bar{\phi}}. \tag{86}$$

B. Solving the wave equations

Eqs. (77) and (78) have the general form

$$\Box Q + 2\mathcal{B}Q_{,0} = 4\pi a^2\mathcal{T}, \quad \mathcal{B} \sim \mathcal{O}(\mathcal{H}), \quad \dot{\mathcal{B}} \sim \mathcal{O}(\mathcal{H}^2). \tag{87}$$

This can be solved by introducing two new functions, $b = b(\eta)$ and $q = q(\eta, x^i)$, such that

$$Q = b^2 q, \tag{88}$$

where $b = b(\eta)$ is *defined* by

$$\dot{b} = \mathcal{B}b, \quad \dot{b} \sim \mathcal{O}(\mathcal{H}), \quad \ddot{b} \sim \mathcal{O}(\mathcal{H}^2). \tag{89}$$

Noticing that, in the linear $\mathcal{O}(\mathcal{H})$ approximation,

$$\Box Q + 2\mathcal{B}Q_{,0} = b\Box(bq), \tag{90}$$

we get the equation

$$\Box(bq) = 4\pi\frac{a^2\mathcal{T}}{b}, \tag{91}$$

whose retarded solution is given by

$$q(\eta, \mathbf{x}) = -\frac{1}{b(\eta)}\int \frac{a^2(\eta')}{b(\eta')}\frac{\mathcal{T}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \eta' = \eta - |\mathbf{x}-\mathbf{x}'|. \tag{92}$$

The corresponding solution to (87) is then given by

$$Q(\eta, \mathbf{x}) = -b(\eta)\int \frac{a^2(\eta')}{b(\eta')}\frac{\mathcal{T}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \eta' = \eta - |\mathbf{x}-\mathbf{x}'|, \tag{93}$$

where, we recall, $\dot{b} = \mathcal{B}b$.

For example, when applied to $Q = b^2 q_{\mu\nu} = \varphi$, the retarded solution (93) takes the form

$$\begin{aligned}\varphi(\eta, \mathbf{x}) &= -\tilde{b}(\eta)\int \frac{a^2(\eta')}{\tilde{b}(\eta')}\frac{2}{3+2\omega(\eta')}\frac{\Lambda(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= -\tilde{b}(\eta)\int \frac{1}{\tilde{b}(\eta')}\frac{2}{3+2\omega(\eta')}\frac{f^{\alpha\beta}\Lambda_{\alpha\beta}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \tilde{\mathcal{B}} \equiv \frac{\dot{\tilde{b}}}{\tilde{b}} = -\mathcal{H} + \frac{\mathcal{F}}{2} - \frac{\omega'\bar{\phi}}{3+2\omega}\mathcal{F}, \quad \omega' \equiv \frac{d\omega}{d\bar{\phi}}.\end{aligned} \tag{94}$$

When applied to $Q_{\mu\nu} = b^2 q_{\mu\nu} = l_{\mu\nu} + \bar{g}_{\mu\nu}\varphi/\bar{\phi}$ introduced in (80) and (78), the retarded solution becomes

$$l_{\mu\nu}(\eta, \mathbf{x}) = \underbrace{b(\eta)\int \frac{\mathcal{S}_{\mu\nu}(\eta', \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{\equiv Q_{\mu\nu}} - \bar{g}_{\mu\nu}\frac{\varphi(\eta, \mathbf{x})}{\bar{\phi}(\eta)}, \quad \mathcal{S}_{\mu\nu} = \frac{-4a^2\Lambda_{\mu\nu}}{b\bar{\phi}}, \quad \mathcal{B} \equiv \frac{\dot{b}}{b} = \mathcal{H} - \frac{\mathcal{F}}{2}. \tag{95}$$

We will use this form of $l_{\mu\nu}$ in Sec. VII C to check the gauge condition.

V. APPENDIX: SOME USEFUL FORMULAS

Given

$$S = \int \mathcal{F}(Q)d^4x, \tag{96}$$

the variational derivative of $\mathcal{F}$ with respect to the variable Q is defined by

$$\frac{\delta\mathcal{F}}{\delta Q} = \frac{\partial\mathcal{F}}{\partial Q} - \frac{\partial}{\partial x^\alpha}\frac{\partial\mathcal{F}}{\partial Q_{,\alpha}} + \frac{\partial^2}{\partial x^\alpha\partial x^\beta}\frac{\partial\mathcal{F}}{\partial Q_{,\alpha\beta}}. \tag{97}$$

It can then be shown that

$$\frac{\delta\mathcal{F}}{\delta Q} = \frac{\partial\mathcal{F}}{\partial Q} - \left[\frac{\partial\mathcal{F}}{\partial Q_{;\alpha}}\right]_{;\alpha} + \left[\frac{\partial\mathcal{F}}{\partial Q_{;\alpha\beta}}\right]_{;\beta\alpha}, \tag{98}$$

and, in the case of $Q = g_{\mu\nu}$ and $\mathcal{F} = \mathcal{F}(g_{\mu\nu}, \Gamma^\alpha_{\mu\nu}, R^\alpha_{\beta\mu\nu})$,

$$\frac{\delta\mathcal{F}}{\delta g_{\mu\nu}} = \frac{\partial\mathcal{F}}{\partial g_{\mu\nu}} - \frac{1}{2}\left(g^{\sigma\nu}\frac{\partial\mathcal{F}}{\partial\Gamma^\sigma_{\mu\alpha}} + g^{\sigma\mu}\frac{\partial\mathcal{F}}{\partial\Gamma^\sigma_{\alpha\nu}} - g^{\sigma\alpha}\frac{\partial\mathcal{F}}{\partial\Gamma^\sigma_{\mu\nu}}\right)_{;\alpha} + \left(g^{\sigma\nu}\frac{\partial\mathcal{F}}{\partial R^\sigma_{\mu\beta\alpha}} + g^{\sigma\mu}\frac{\partial\mathcal{F}}{\partial R^\sigma_{\alpha\beta\nu}} - g^{\sigma\alpha}\frac{\partial\mathcal{F}}{\partial R^\sigma_{\mu\beta\nu}}\right)_{;\beta\alpha}. \tag{99}$$

Now, the full metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varkappa_{\mu\nu}, \quad g^{\alpha\beta} = \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta}, \quad \delta g^{\alpha\beta} \approx -\varkappa^{\alpha\beta} + \varkappa^{\alpha}{}_{\nu}\varkappa^{\nu\beta}, \quad \varkappa^{\alpha}{}_{\nu} = \bar{g}^{\alpha\mu}\varkappa_{\mu\nu}, \quad \varkappa^{\nu\beta} = \bar{g}^{\nu\mu}\varkappa_{\mu\sigma}\bar{g}^{\sigma\beta}. \tag{100}$$

The raised Gothic metric is

$$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta}, \quad \bar{\mathfrak{g}}^{\alpha\beta} \equiv \sqrt{-\bar{g}}\bar{g}^{\alpha\beta}, \quad \mathfrak{g}^{\alpha\beta} = \bar{\mathfrak{g}}^{\alpha\beta} + \mathfrak{h}^{\alpha\beta}, \quad \mathfrak{h}^{\alpha\beta} = \sqrt{-\bar{g}}l^{\alpha\beta}. \tag{101}$$

Given

$$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta}, \tag{102}$$

the relationship between the determinants of $g_{\alpha\beta}$ and $\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta}/\sqrt{-g}$,

$$g = \mathfrak{g}^{-1}, \quad \mathfrak{g} \equiv \det\left[\mathfrak{g}_{\alpha\beta}\right], \tag{103}$$

and using

$$\frac{\partial\left(\sqrt{-g}\right)}{\partial g_{\mu\nu}} = +\frac{1}{2}\sqrt{-g}g^{\mu\nu}, \quad \frac{\partial\left(\sqrt{-g}\right)}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}, \quad \frac{\partial g^{\lambda\kappa}}{\partial g_{\mu\nu}} = -g^{\lambda\mu}g^{\kappa\nu}, \tag{104}$$

we have,

$$\begin{aligned}
\frac{\partial g^{\alpha\beta}}{\partial \mathfrak{g}^{\mu\nu}} &= \frac{\partial\left(\frac{1}{\sqrt{-g}}\mathfrak{g}^{\alpha\beta}\right)}{\partial \mathfrak{g}^{\mu\nu}} \\
&= \frac{\partial\left(\sqrt{-\mathfrak{g}}\mathfrak{g}^{\alpha\beta}\right)}{\partial \mathfrak{g}^{\mu\nu}} \\
&= -\frac{1}{2}\sqrt{-\mathfrak{g}}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\alpha\beta} + \sqrt{-\mathfrak{g}}\frac{\partial\left(\mathfrak{g}^{\alpha\beta}\right)}{\partial \mathfrak{g}^{\mu\nu}} \\
&= -\frac{1}{2}\sqrt{-\mathfrak{g}}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\alpha\beta} + \frac{1}{2}\sqrt{-\mathfrak{g}}\left(\delta^{\alpha}_{\mu}\delta^{\beta\nu} + \delta^{\alpha}_{\nu}\delta^{\beta\mu}\right) \\
&= \frac{1}{2\sqrt{-g}}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu} - g^{\alpha\beta}g_{\mu\nu}\right),
\end{aligned} \tag{105}$$

and, similarly,

$$\frac{\partial g_{\alpha\beta}}{\partial \mathfrak{g}^{\mu\nu}} = -\frac{1}{2\sqrt{-g}}\left(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu}\right). \tag{106}$$

It then follows that

$$\frac{\partial}{\partial \mathfrak{g}^{\mu\nu}}\left(g_{\alpha\beta}g_{\rho\sigma}\right) = -\frac{1}{2\sqrt{-g}}\left[g_{\alpha\beta}\left(g_{\rho\mu}g_{\sigma\nu} + g_{\rho\nu}g_{\sigma\mu} - g_{\rho\sigma}g_{\mu\nu}\right) + \left(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu}\right)g_{\rho\sigma}\right]. \tag{107}$$

Then,

$$l^{\alpha\beta} = -\varkappa^{\alpha\beta} + \frac{1}{2}\bar{g}^{\alpha\beta}\varkappa + \varkappa^{\mu(\alpha}\varkappa^{\beta)}{}_{\mu} - \frac{1}{2}\varkappa^{\alpha\beta}\varkappa - \frac{1}{4}\bar{g}^{\alpha\beta}\left(\varkappa^{\mu\nu}\varkappa_{\mu\nu} - \frac{1}{2}\varkappa^{2}\right), \tag{108}$$

and

$$\varkappa_{\alpha\beta} = -l_{\alpha\beta} + \frac{1}{2}\bar{g}_{\alpha\beta}l + l^{\mu}{}_{(\alpha}l_{\beta)\mu} - \frac{1}{2}l_{\alpha\beta}l - \frac{1}{4}\bar{g}_{\alpha\beta}\left(l^{\mu\nu}l_{\mu\nu} - \frac{1}{2}l^{2}\right), \tag{109}$$

where

$$\varkappa \equiv \varkappa^{\alpha}{}_{\alpha} = \bar{g}^{\alpha\beta}\varkappa_{\alpha\beta}, \tag{110}$$

so in linear order in $\varkappa^{\alpha\beta}$ we get

$$l = \varkappa, \quad l_{\mu\nu} = -\varkappa_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\varkappa, \quad \varkappa_{\mu\nu} = -l_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}l, \quad \varkappa^{\alpha\beta} = -l^{\alpha\beta} + \frac{1}{2}\bar{g}^{\alpha\beta}l. \tag{111}$$

For additional details the reader may consult Ref. [1].

VI. APPENDIX: WORKING OUT $l_{\mu\nu}{}^{|\alpha}{}_{|\alpha}$ TO LINEAR ORDER IN $\mathcal{H}$

We first notice that for

$$B^\alpha = -\frac{2\mathcal{H}}{a} l^{\alpha\beta}\bar{u}_\beta, \tag{112}$$

we have

$$B^\alpha{}_{|\alpha} = -\frac{2\mathcal{H}}{a} l^{\alpha\beta}{}_{|\alpha}\bar{u}_\beta + O(\mathcal{H}^2) = -\frac{2\mathcal{H}}{a}\bar{u}^\alpha A_\alpha + O(\mathcal{H}^2) = +\frac{2\mathcal{H}}{a}\bar{u}^\alpha\frac{\varphi_{|\alpha}}{\bar{\phi}} + O(\mathcal{H}^2), \tag{113}$$

where the gauge condition (67) has been used. Additionally, we have:

$$\begin{aligned}
l_{\alpha\beta}{}^{|\mu}{}_{|\mu} &= \bar{g}^{\mu\nu} l_{\alpha\beta|\mu\nu} \\
&= \bar{g}^{\mu\nu}\left[\left(l_{\alpha\beta|\mu}\right)_{,\nu} - \bar{\Gamma}^\rho_{\alpha\nu} l_{\rho\beta|\mu} - \bar{\Gamma}^\rho_{\beta\nu} l_{\alpha\rho|\mu} - \bar{\Gamma}^\rho_{\mu\nu} l_{\alpha\beta|\rho}\right] \\
&= \bar{g}^{\mu\nu}\left[\left(l_{\alpha\beta,\mu} - \bar{\Gamma}^\rho_{\alpha\mu} l_{\rho\beta} - \bar{\Gamma}^\rho_{\beta\mu} l_{\alpha\rho}\right)_{,\nu} - \bar{\Gamma}^\rho_{\alpha\nu} l_{\rho\beta|\mu} - \bar{\Gamma}^\rho_{\beta\nu} l_{\alpha\rho|\mu} - \bar{\Gamma}^\rho_{\mu\nu} l_{\alpha\beta|\rho}\right] \\
&= \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \bar{g}^{\mu\nu}\left[\frac{\mathcal{H}}{a}\left(\delta^\rho_\alpha\bar{u}_\mu + \delta^\rho_\mu\bar{u}_\alpha - \bar{u}^\rho\bar{g}_{\alpha\mu}\right) l_{\rho\beta} + \frac{\mathcal{H}}{a}\left(\delta^\rho_\beta\bar{u}_\mu + \delta^\rho_\mu\bar{u}_\beta - \bar{u}^\rho\bar{g}_{\beta\mu}\right) l_{\alpha\rho}\right]_{,\nu} \\
&\quad + \frac{\mathcal{H}}{a}\bar{g}^{\mu\nu}\left[\left(\delta^\rho_\alpha\bar{u}_\nu + \delta^\rho_\nu\bar{u}_\alpha - \bar{u}^\rho\bar{g}_{\alpha\nu}\right) l_{\rho\beta|\mu} + \left(\delta^\rho_\beta\bar{u}_\nu + \delta^\rho_\nu\bar{u}_\beta - \bar{u}^\rho\bar{g}_{\beta\nu}\right) l_{\alpha\rho|\mu} + \left(\delta^\rho_\mu\bar{u}_\nu + \delta^\rho_\nu\bar{u}_\mu - \bar{u}^\rho\bar{g}_{\mu\nu}\right) l_{\alpha\beta|\rho}\right] \\
&\approx \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\left[\bar{u}^\mu l_{\alpha\beta,\mu} - \bar{u}^\mu l_{\mu\beta,\alpha} - \bar{u}^\mu l_{\alpha\mu,\beta} + \bar{g}^{\mu\nu}\bar{u}_\alpha l_{\mu\beta,\nu} + \bar{g}^{\mu\nu}\bar{u}_\beta l_{\alpha\mu,\nu}\right] \\
&\approx \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\left[\bar{u}^\mu l_{\alpha\beta,\mu} - \bar{u}^\mu l_{\mu\beta|\alpha} - \bar{u}^\mu l_{\alpha\mu|\beta} + \bar{u}_\alpha l_{\beta\mu}{}^{|\mu} + \bar{u}_\beta l_{\alpha\mu}{}^{|\mu}\right] \\
&\approx \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\bar{u}^\mu\partial_\mu l_{\alpha\beta} - \left(\frac{2\mathcal{H}}{a}\bar{u}^\mu l_{\mu\beta}\right)_{|\alpha} - \left(\frac{2\mathcal{H}}{a}\bar{u}^\mu l_{\mu\alpha}\right)_{|\beta} + \frac{2\mathcal{H}}{a}\bar{u}_\alpha l_{\beta\mu}{}^{|\mu} + \frac{2\mathcal{H}}{a}\bar{u}_\beta l_{\alpha\mu}{}^{|\mu} \\
&= \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\bar{u}^\mu\partial_\mu l_{\alpha\beta} - \left(\frac{2\mathcal{H}}{a}\bar{u}^\mu l_{\mu\beta}\right)_{|\alpha} - \left(\frac{2\mathcal{H}}{a}\bar{u}^\mu l_{\mu\alpha}\right)_{|\beta} + \frac{2\mathcal{H}}{a}\bar{u}_\alpha A_\beta + \frac{2\mathcal{H}}{a}\bar{u}_\beta A_\alpha \\
&= \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\bar{u}^\mu\partial_\mu l_{\alpha\beta} + B_{\beta|\alpha} + B_{\alpha|\beta} - \frac{2\mathcal{H}}{a}\bar{u}_\alpha\frac{\varphi_{|\beta}}{\bar{\phi}} - \frac{2\mathcal{H}}{a}\bar{u}_\beta\frac{\varphi_{|\alpha}}{\bar{\phi}}.
\end{aligned} \tag{114}$$

Then, using (113), we get

$$\begin{aligned}
l_{\alpha\beta}{}^{|\mu}{}_{|\mu} + \bar{g}_{\alpha\beta}B^\mu{}_{|\mu} - B_{\alpha|\beta} - B_{\beta|\alpha} &= \bar{g}^{\mu\nu} l_{\alpha\beta,\mu\nu} + \frac{2\mathcal{H}}{a}\bar{u}^\mu\partial_\mu l_{\alpha\beta} + \bar{g}_{\alpha\beta}\frac{2\mathcal{H}}{a}\bar{u}^\mu\frac{\varphi_{|\mu}}{\bar{\phi}} - \frac{2\mathcal{H}}{a}\left(\bar{u}_\alpha\frac{\varphi_{|\beta}}{\bar{\phi}} + \bar{u}_\beta\frac{\varphi_{|\alpha}}{\bar{\phi}}\right) \\
&= \frac{1}{a^2}\left(\Box l_{\alpha\beta} + 2\mathcal{H} l_{\alpha\beta,0}\right) + \frac{2\mathcal{H}}{a}\left(\bar{g}_{\alpha\beta}\bar{u}^\mu\frac{\varphi_{|\mu}}{\bar{\phi}} - \bar{u}_\alpha\frac{\varphi_{|\beta}}{\bar{\phi}} - \bar{u}_\beta\frac{\varphi_{|\alpha}}{\bar{\phi}}\right).
\end{aligned} \tag{115}$$

VII. APPENDIX: CHECKING THE GAUGE CONDITION

A. Classical electrodynamics

As a warm-up exercise, let us recall how the gauge condition is checked in classical electrodynamics. In that case, the retarded solution for the vector potential A^μ is given by

$$A^\mu(t,\mathbf{x}) = \int\frac{j^\mu(t',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad t' = t - |\mathbf{x}-\mathbf{x}'|. \tag{116}$$

Using the notation

$$j'^\mu \equiv j^\mu(t',\mathbf{x}'), \quad t' = t - R, \quad R \equiv |\mathbf{x}-\mathbf{x}'|, \quad \partial'_\mu \equiv \partial/\partial x'^\mu, \tag{117}$$

we get

$$\begin{aligned}
\partial_\mu A^\mu(t,\mathbf{x}) &= \partial_\mu\left(\int \frac{j'^\mu}{R}d^3x'\right) = \int \partial_\mu\left(\frac{j'^\mu}{R}\right)d^3x' \\
&= \int\left[\frac{1}{R}\partial_\mu j'^\mu + j'^\mu\partial_\mu\left(\frac{1}{R}\right)\right]d^3x' \\
&= \int\left[\frac{1}{R}\left(\partial_0 j'^0 + \partial_k j'^k\right) + j'^k\partial_k\left(\frac{1}{R}\right)\right]d^3x' \\
&= \int\left\{\frac{1}{R}\left[\partial'_0 j'^0 + \left(\partial'_0 j'^k\right)(\partial_k t')\right] + j'^k\partial_k\left(\frac{1}{R}\right)\right\}d^3x' \\
&= \int\left\{\frac{1}{R}\left[\partial'_0 j'^0 + \left(\partial'_0 j'^k\right)(-\partial_k R)\right] - j'^k\partial'_k\left(\frac{1}{R}\right)\right\}d^3x' \\
&= \int\left\{\frac{1}{R}\left[\partial'_0 j'^0 + \left(\partial'_0 j'^k\right)(\partial'_k R)\right] - j'^k\partial'_k\left(\frac{1}{R}\right)\right\}d^3x'. \qquad (118)
\end{aligned}$$

We now notice that

$$\partial'_k j'^k = \left(\partial'_k j'^k\right)_{t'=\text{const}} + \left(\partial'_0 j'^k\right)(-\partial'_k R), \qquad (119)$$

and thus

$$\left(\partial'_0 j'^k\right)(\partial'_k R) = \left(\partial'_k j'^k\right)_{t'=\text{const}} - \partial'_k j'^k. \qquad (120)$$

Substituting (120) into (118) finally gives

$$\begin{aligned}
\partial_\mu A^\mu(t,\mathbf{x}) &= \int\left\{\frac{1}{R}\left[\partial'_0 j'^0 + \left(\partial'_k j'^k\right)_{t'=\text{const}}\right] - \left[\frac{1}{R}\partial'_k j'^k + j'^k\partial'_k\left(\frac{1}{R}\right)\right]\right\}d^3x' \\
&= \int\frac{1}{R}\underbrace{\left[\partial'_0 j'^0 + \left(\partial'_k j'^k\right)_{t'=\text{const}}\right]}_{=0,\text{ continuity}}d^3x' - \underbrace{\int\partial'_k\left(\frac{j'^k}{R}\right)d^3x'}_{=0,\text{ divergence}} \\
&= 0, \qquad (121)
\end{aligned}$$

as expected.

B. "Celestial ephemerides" solution

Now let us verify the gauge,

$$B^\alpha \equiv -\frac{2\mathcal{H}}{a}l^{\alpha\beta}\bar{u}_\beta, \qquad (122)$$

used in the "Celestial ephemerides" paper [3].

First, notice that in conformal coordinates, for any symmetric tensor $l^{\mu\nu}$,

$$\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta|\nu} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\left(l_{\alpha\beta,\nu} - \Gamma^\kappa_{\alpha\nu}l_{\kappa\beta} - \Gamma^\kappa_{\beta\nu}l_{\alpha\kappa}\right) \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} + \bar{g}^{\mu\alpha}\frac{\mathcal{H}}{a}\left(\delta^\kappa_\alpha\bar{u}_\nu + \delta^\kappa_\nu\bar{u}_\alpha - \bar{u}^\kappa\bar{g}_{\alpha\nu}\right)l_\kappa{}^\nu + \bar{g}^{\nu\beta}\frac{\mathcal{H}}{a}\left(\delta^\kappa_\beta\bar{u}_\nu + \delta^\kappa_\nu\bar{u}_\beta - \bar{u}^\kappa\bar{g}_{\beta\nu}\right)l^\mu{}_\kappa \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} + \frac{\mathcal{H}}{a}\left(\bar{g}^{\mu\kappa}\bar{u}_\nu + \delta^\kappa_\nu\bar{u}^\mu - \bar{u}^\kappa\delta^\mu_\nu\right)l_\kappa{}^\nu + \frac{\mathcal{H}}{a}\left(\bar{g}^{\nu\kappa}\bar{u}_\nu + \bar{u}^\kappa - 4\bar{u}^\kappa\right)l^\mu{}_\kappa \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} + \frac{\mathcal{H}}{a}l\bar{u}^\mu - \frac{2\mathcal{H}}{a}l^{\mu\nu}\bar{u}_\nu \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} + \frac{\mathcal{H}}{a^2}l\delta^\mu_0 + 2\mathcal{H}l^{\mu 0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0}. \qquad (123)
\end{aligned}$$

Next, the retarded solution, $l^{\mu\nu}$, used in the "Celestial ephemerides" paper is

$$l_{\mu\nu}(t,\mathbf{x}) = -4a(\eta)\int \frac{a(\eta')\mathcal{T}_{\mu\nu}(\eta',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \quad \eta' = \eta - |\mathbf{x}-\mathbf{x}'|, \tag{124}$$

where $\mathcal{T}_{\mu\nu}$ is the stress-energy tensor of the matter perturbation. We have,

$$\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(-4a\int\frac{a'\mathcal{T}'_{\alpha\beta}}{R}d^3x'\right) - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(\partial_\nu a)\left(-4\int\frac{a'\mathcal{T}'_{\alpha\beta}}{R}d^3x'\right) + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(-4a)\int\partial_\nu\left(\frac{a'\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{0\beta}\mathcal{H}l_{\alpha\beta} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(-4a)\int\partial_\nu\left(\frac{a'\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0} \\
&= \mathcal{H}l^{\mu 0} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(-4a)\int\partial_\nu\left(\frac{a'\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(-4a)\int\partial_\nu\left(\frac{a'\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu 0} + 3\mathcal{H}l^{\mu 0} \\
&= \frac{1}{a^4}(-4a)\int\partial_\nu\left(\frac{a'f^{\mu\alpha}f^{\nu\beta}\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu 0} + 3\mathcal{H}l^{\mu 0}.
\end{aligned} \tag{125}$$

Now performing a few steps analogous to those in electrodynamics (see Eq. (118)) gives

$$\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= -\mathcal{H}l\bar{g}^{\mu 0} + 3\mathcal{H}l^{\mu 0} \\
&\quad + \frac{1}{a^4}(-4a)\int\frac{1}{R}\left\{\partial'_0\left(a'f^{\mu\alpha}f^{0\beta}\mathcal{T}'_{\alpha\beta}\right) + \left[\partial'_k\left(a'f^{\mu\alpha}f^{k\beta}\mathcal{T}'_{\alpha\beta}\right)\right]_{t'=\text{const}}\right\}d^3x' - \frac{1}{a^4}(-4a)\underbrace{\int\partial'_k\left(\frac{a'f^{\mu\alpha}f^{k\beta}\mathcal{T}'_{\alpha\beta}}{R}\right)d^3x'}_{=0,\text{ divergence}} \\
&= -\mathcal{H}l\bar{g}^{\mu 0} + 3\mathcal{H}l^{\mu 0} + \frac{1}{a^4}(-4a)\int\frac{1}{R}\underbrace{\left\{\partial'_0\left(a'f^{\mu\alpha}f^{0\beta}\mathcal{T}'_{\alpha\beta}\right) + \left[\partial'_k\left(a'f^{\mu\alpha}f^{k\beta}\mathcal{T}'_{\alpha\beta}\right)\right]_{t'=\text{const}}\right\}}_{\equiv I'^{\mu}}d^3x'.
\end{aligned} \tag{126}$$

Conservation of $\mathcal{T}^{\mu\nu}$,

$$\partial_\beta\left(\sqrt{-\bar{g}}\mathcal{T}^{\alpha\beta}\right) + \sqrt{-\bar{g}}\Gamma^\alpha{}_{\beta\gamma}\mathcal{T}^{\beta\gamma} = 0, \tag{127}$$

gives, in conformal coordinates,

$$\partial_0\left(a\mathcal{T}_{00}\right) - \partial_j\left(a\mathcal{T}_{0j}\right) = -\mathcal{H}\left(a\mathcal{T}_{kk}\right), \tag{128}$$

$$\partial_0\left(a\mathcal{T}_{i0}\right) - \partial_j\left(a\mathcal{T}_{ij}\right) = -\mathcal{H}\left(a\mathcal{T}_{i0}\right). \tag{129}$$

Then for the 0-th component we get

$$\begin{aligned}
I^0 &= \partial_0\left(af^{0\alpha}f^{0\beta}\mathcal{T}_{\alpha\beta}\right) + \partial_k\left(af^{0\alpha}f^{k\beta}\mathcal{T}_{\alpha\beta}\right) \\
&= \partial_0\left(af^{00}f^{00}\mathcal{T}_{00}\right) + \partial_k\left(af^{00}f^{km}\mathcal{T}_{0m}\right) \\
&= \partial_0\left(a\mathcal{T}_{00}\right) - \partial_k\left(a\mathcal{T}_{0k}\right) \\
&= -\mathcal{H}\left(a\mathcal{T}_{kk}\right), \\
&= -\mathcal{H}\left(af^{ik}\mathcal{T}_{ik}\right),
\end{aligned} \tag{130}$$

and thus,

$$\begin{aligned}
l^{0\nu}{}_{|\nu} &= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} - \frac{f^{ik}}{a^4}(-4a)\underbrace{\int \frac{\mathcal{H}'}{R}\,(a'\mathcal{T}'_{ik})\,d^3x'}_{\text{treat } \mathcal{H}' \text{ as constant}} \\
&\approx -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} - \mathcal{H}\frac{f^{ik}}{a^4}(-4a)\int \frac{1}{R}\,(a'\mathcal{T}'{}_{ik})\,d^3x' \\
&= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}(-4a)\int \frac{1}{R}\,(a'\mathcal{T}'{}_{ik})\,d^3x' \\
&= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}l_{ik} \\
&= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} + \mathcal{H}\bar{g}^{ik}l_{ik}\bar{g}^{00} \\
&= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} + \mathcal{H}\bar{g}^{ik}l_{ik}\bar{g}^{00} + \left(\mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00} - \mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00}\right) \\
&= -\mathcal{H}l\bar{g}^{00} + 3\mathcal{H}l^{00} + \mathcal{H}l\bar{g}^{00} - \mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00} \\
&= 2\mathcal{H}l^{00}. \qquad (131)
\end{aligned}$$

For the i-th component,

$$\begin{aligned}
I^i &= \partial_0\left(af^{i\alpha}f^{0\beta}\mathcal{T}_{\alpha\beta}\right) + \partial_k\left(af^{i\alpha}f^{k\beta}\mathcal{T}_{\alpha\beta}\right) \\
&= \partial_0\left(af^{ij}f^{00}\mathcal{T}_{j0}\right) + \partial_k\left(af^{ij}f^{km}\mathcal{T}_{jm}\right) \\
&= -\partial_0\left(a\mathcal{T}_{i0}\right) + \partial_k\left(a\mathcal{T}_{ik}\right) \\
&= +\mathcal{H}\left(a\mathcal{T}_{i0}\right), \qquad (132)
\end{aligned}$$

and thus

$$\begin{aligned}
l^{i\nu}{}_{|\nu} &= 3\mathcal{H}l^{i0} + \frac{1}{a^4}(-4a)\underbrace{\int \frac{\mathcal{H}'}{R}\,(a'\mathcal{T}'_{i0})\,d^3x'}_{\text{treat } \mathcal{H}' \text{ as constant}} \\
&\approx 3\mathcal{H}l^{i0} + \mathcal{H}\frac{1}{a^4}(-4a)\int \frac{1}{R}\,(a'\mathcal{T}'{}_{i0})\,d^3x' \\
&= 3\mathcal{H}l^{i0} + \mathcal{H}\frac{1}{a^4}l_{i0} \\
&= 3\mathcal{H}l^{i0} + \mathcal{H}\frac{1}{a^4}\bar{g}_{i\mu}\bar{g}_{0\nu}l^{\mu\nu} \\
&= 3\mathcal{H}l^{i0} + \mathcal{H}f_{i\mu}f_{0\nu}l^{\mu\nu} \\
&= 3\mathcal{H}l^{i0} + \mathcal{H}f_{ik}f_{00}l^{k0} \\
&= 3\mathcal{H}l^{i0} - \mathcal{H}l^{i0} \\
&= 2\mathcal{H}l^{i0}. \qquad (133)
\end{aligned}$$

This shows that in conformal coordinates

$$l^{\mu\nu}{}_{|\nu} = 2\mathcal{H}l^{\mu 0}, \qquad (134)$$

as required.

C. Our scalar-tensor theory solution

We will again use the primed notation,

$$\mathcal{S}'_{\mu\nu} \equiv \mathcal{S}_{\mu\nu}(t', \mathbf{x}'), \quad t' = t - R, \quad R \equiv |\mathbf{x} - \mathbf{x}'|, \quad \partial'_\mu \equiv \partial/\partial x'^\mu, \qquad (135)$$

We first write down (123), which is a general result for any symmetric tensor expressed in conformal coordinates,

$$l^{\mu\nu}{}_{|\nu} = \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} - \mathcal{H}l\bar{g}^{\mu 0} + 2\mathcal{H}l^{\mu 0}. \qquad (136)$$

Next, for our retarded solution [5], $l^{\mu\nu}$, as found in (95),

$$l_{\mu\nu}(\eta,\mathbf{x}) = \underbrace{b(\eta)\int\frac{\mathcal{S}_{\mu\nu}(\eta',\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{\equiv Q_{\mu\nu}} - \bar{g}_{\mu\nu}\frac{\varphi(\eta,\mathbf{x})}{\bar{\phi}(\eta)}, \quad \mathcal{S}_{\mu\nu} = \frac{-4a^2\Lambda_{\mu\nu}}{b\bar{\phi}}, \quad \mathcal{B}\equiv\frac{\dot{b}}{b} = \mathcal{H}-\frac{\mathcal{F}}{2}, \tag{137}$$

we get

$$\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}l_{\alpha\beta,\nu} - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(b\int\frac{\mathcal{S}'_{\alpha\beta}}{R}d^3x' - \bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}(\partial_\nu b)\left(\int\frac{\mathcal{S}'_{\alpha\beta}}{R}d^3x'\right) + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \bar{g}^{\mu\alpha}\bar{g}^{0\beta}\mathcal{B}Q_{\alpha\beta} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \mathcal{B}Q^{\mu0} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \mathcal{B}\left(l^{\mu0} + \bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}}\right) + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \mathcal{B}l^{\mu0} + \mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + 2\mathcal{H}l^{\mu0} \\
&= \mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}b\int\partial_\nu\left(\frac{\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + (2\mathcal{H}+\mathcal{B})\,l^{\mu0} \\
&= \mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) + \frac{1}{a^4}b\int\partial_\nu\left(\frac{f^{\mu\alpha}f^{\nu\beta}\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x' - \mathcal{H}l\bar{g}^{\mu0} + (2\mathcal{H}+\mathcal{B})\,l^{\mu0}.
\end{aligned} \tag{138}$$

A few additional steps analogous to those in electrodynamics (see discussion following Eq. (118)) give

$$\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= \mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + (2\mathcal{H}+\mathcal{B})\,l^{\mu0} \\
&\quad + \frac{1}{a^4}b\int\frac{1}{R}\left\{\partial'_0\left(f^{\mu\alpha}f^{0\beta}\mathcal{S}'_{\alpha\beta}\right) + \left[\partial'_k\left(f^{\mu\alpha}f^{k\beta}\mathcal{S}'_{\alpha\beta}\right)\right]_{t'=\text{const}}\right\}d^3x' - \frac{1}{a^4}b\underbrace{\int\partial'_k\left(\frac{f^{\mu\alpha}f^{k\beta}\mathcal{S}'_{\alpha\beta}}{R}\right)d^3x'}_{=0,\text{ divergence}} \\
&= \mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}l\bar{g}^{\mu0} + (2\mathcal{H}+\mathcal{B})\,l^{\mu0} \\
&\quad + \frac{1}{a^4}b\int\frac{1}{R}\underbrace{\left\{\partial'_0\left(f^{\mu\alpha}f^{0\beta}\mathcal{S}'_{\alpha\beta}\right) + \left[\partial'_k\left(f^{\mu\alpha}f^{k\beta}\mathcal{S}'_{\alpha\beta}\right)\right]_{t'=\text{const}}\right\}}_{\equiv I'^{\mu}}d^3x'.
\end{aligned} \tag{139}$$

Notice that

$$\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) = \bar{g}^{\mu0}(2\mathcal{H}-\mathcal{F})\frac{\varphi}{\bar{\phi}} + \frac{\varphi^{|\mu}}{\bar{\phi}}, \tag{140}$$

so

$$\begin{aligned}
\mathcal{B}\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\partial_\nu\left(\bar{g}_{\alpha\beta}\frac{\varphi}{\bar{\phi}}\right) &= \bar{g}^{\mu0}\left(\mathcal{H}-\frac{\mathcal{F}}{2}\right)\frac{\varphi}{\bar{\phi}} - \bar{g}^{\mu0}(2\mathcal{H}-\mathcal{F})\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|\mu}}{\bar{\phi}} \\
&= \left(-\mathcal{H}+\frac{\mathcal{F}}{2}\right)\bar{g}^{\mu0}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|\mu}}{\bar{\phi}},
\end{aligned} \tag{141}$$

and thus

$$
\begin{aligned}
l^{\mu\nu}{}_{|\nu} &= \left(-\mathcal{H}+\frac{\mathcal{F}}{2}\right)\bar{g}^{\mu 0}\frac{\varphi}{\bar{\phi}}-\frac{\varphi^{|\mu}}{\bar{\phi}}-\mathcal{H}l\bar{g}^{\mu 0}+(2\mathcal{H}+\mathcal{B})\,l^{\mu 0} \\
&\quad +\frac{1}{a^4}b\int\frac{1}{R}\underbrace{\left\{\partial'_0\left(f^{\mu\alpha}f^{0\beta}\mathcal{S}'_{\alpha\beta}\right)+\left[\partial'_k\left(f^{\mu\alpha}f^{k\beta}\mathcal{S}'_{\alpha\beta}\right)\right]_{t'=\text{const}}\right\}}_{\equiv I'^{\mu}}d^3x'.
\end{aligned}
\tag{142}
$$

For the 0-th component of I^μ we get

$$
\begin{aligned}
I^0 &= \partial_0\left(f^{0\alpha}f^{0\beta}\mathcal{S}_{\alpha\beta}\right)+\partial_k\left(f^{0\alpha}f^{k\beta}\mathcal{S}_{\alpha\beta}\right) \\
&= \partial_0\left(f^{00}f^{00}\mathcal{S}_{00}\right)+\partial_k\left(f^{00}f^{km}\mathcal{S}_{0m}\right) \\
&= \partial_0\left(\mathcal{S}_{00}\right)-\partial_k\left(\mathcal{S}_{0k}\right) \\
&= \partial_0\left(\frac{-4a^2\Lambda_{00}}{b\bar{\phi}}\right)-\partial_k\left(\frac{-4a^2\Lambda_{0k}}{b\bar{\phi}}\right) \\
&= \partial_0\left(\frac{-4a\left(a\Lambda_{00}\right)}{b\bar{\phi}}\right)-\partial_k\left(\frac{-4a\left(a\Lambda_{0k}\right)}{b\bar{\phi}}\right) \\
&= \partial_0\left(\frac{-4a}{b\bar{\phi}}\right)\left(a\Lambda_{00}\right)-\underbrace{\partial_k\left(\frac{-4a}{b\bar{\phi}}\right)}_{=0}\left(a\Lambda_{0k}\right)+\frac{-4a}{b\bar{\phi}}\partial_0\left(a\Lambda_{00}\right)-\frac{-4a}{b\bar{\phi}}\partial_k\left(a\Lambda_{0k}\right) \\
&= \mathcal{H}\frac{-4a^2\Lambda_{00}}{b\bar{\phi}}-\mathcal{B}\frac{-4a^2\Lambda_{00}}{b\bar{\phi}}-\mathcal{F}\frac{-4a^2\Lambda_{00}}{b\bar{\phi}}+\frac{-4a}{b\bar{\phi}}\left\{\partial_0\left(a\Lambda_{00}\right)-\partial_k\left(a\Lambda_{0k}\right)\right\} \\
&= \left(\mathcal{H}-\mathcal{B}-\mathcal{F}\right)\mathcal{S}_{00}+\frac{-4a}{b\bar{\phi}}\left\{\partial_0\left[a\left(\Lambda_{00}+\Lambda^{\varphi}_{00}\right)\right]-\partial_k\left[a\left(\Lambda_{0k}+\Lambda^{\varphi}_{0k}\right)\right]\right\} \\
&= -\frac{\mathcal{F}}{2}\mathcal{S}_{00}+\frac{-4a}{b\bar{\phi}}\left\{\partial_0\left(a\Lambda_{00}\right)-\partial_k\left(a\Lambda_{0k}\right)\right\}.
\end{aligned}
\tag{143}
$$

For the i-th component of I^μ we similarly get

$$
\begin{aligned}
I^i &= \partial_0\left(f^{i\alpha}f^{0\beta}\mathcal{S}_{\alpha\beta}\right)+\partial_k\left(f^{i\alpha}f^{k\beta}\mathcal{S}_{\alpha\beta}\right) \\
&= \partial_0\left(f^{ij}f^{00}\mathcal{S}_{j0}\right)+\partial_k\left(f^{ij}f^{km}\mathcal{S}_{jm}\right) \\
&= -\partial_0\left(\mathcal{S}_{i0}\right)+\partial_k\left(\mathcal{S}_{ik}\right) \\
&= -\partial_0\left(\frac{-4a^2\Lambda_{i0}}{b\bar{\phi}}\right)+\partial_k\left(\frac{-4a^2\Lambda_{ik}}{b\bar{\phi}}\right) \\
&= -\mathcal{H}\left(\frac{-4a^2\Lambda_{i0}}{b\bar{\phi}}\right)+\mathcal{B}\left(\frac{-4a^2\Lambda_{i0}}{b\bar{\phi}}\right)+\mathcal{F}\left(\frac{-4a^2\Lambda_{i0}}{b\bar{\phi}}\right)-\frac{-4a}{b\bar{\phi}}\left\{\partial_0\left(a\Lambda_{i0}\right)-\partial_k\left(a\Lambda_{ik}\right)\right\} \\
&= -\left(\mathcal{H}-\mathcal{B}-\mathcal{F}\right)\mathcal{S}_{i0}-\frac{-4a}{b\bar{\phi}}\left\{\left(a\Lambda_{i0}\right)-\partial_k\left(a\Lambda_{ik}\right)\right\} \\
&= +\frac{\mathcal{F}}{2}\mathcal{S}_{i0}-\frac{-4a}{b\bar{\phi}}\left\{\left(a\Lambda_{i0}\right)-\partial_k\left(a\Lambda_{ik}\right)\right\}.
\end{aligned}
\tag{144}
$$

To find out what $\partial_0\left(a\Lambda_{00}\right)-\partial_k\left(a\Lambda_{0k}\right)$ and $\partial_0\left(a\Lambda_{i0}\right)-\partial_k\left(a\Lambda_{ik}\right)$ are we need to do a few additional calculations. First, by taking the covariant divergence of (63) we get, in the linear Hubble approximation,

$$
8\pi\Lambda^{\mu\nu}{}_{|\nu}=\frac{1}{2}\bar{\phi}^{|\mu}\underbrace{\left(l^{\nu\beta}{}_{|\nu\beta}+\frac{1}{2}l^{|\nu}{}_{|\nu}-\frac{2\omega}{\bar{\phi}}\varphi^{|\nu}{}_{|\nu}\right)}_{=\mathcal{O}(\mathcal{H}),\text{ by (62)}}=0.
\tag{145}
$$

On the other hand, for any symmetric tensor $s^{\mu\nu}$ we have

$$
s^{\mu\nu}{}_{|\nu}=\bar{g}^{\mu\alpha}\left[\frac{1}{\sqrt{-\bar{g}}}\partial_\nu\left(\sqrt{-\bar{g}}\bar{g}^{\beta\nu}s_{\alpha\beta}\right)-\frac{1}{2}\left(\partial_\alpha\bar{g}_{\beta\nu}\right)s^{\beta\nu}\right],
\tag{146}
$$

which in conformal coordinates takes the form

$$\partial_0 (as_{00}) - \partial_j (as_{0j}) = -\mathcal{H} (as_{kk}) + a^5 s^{0\nu}{}_{|\nu}, \tag{147}$$

$$\partial_0 (as_{i0}) - \partial_j (as_{ij}) = -\mathcal{H} (as_{i0}) - a^5 s^{i\nu}{}_{|\nu}. \tag{148}$$

Applying this to $\Lambda^{\mu\nu}$ gives,

$$\partial_0 (a\Lambda_{00}) - \partial_j (a\Lambda_{0j}) = -\mathcal{H} (a\Lambda_{kk}), \tag{149}$$

$$\partial_0 (a\Lambda_{i0}) - \partial_j (a\Lambda_{ij}) = -\mathcal{H} (a\Lambda_{i0}). \tag{150}$$

Therefore, for the 0-th component of I^μ we have

$$\begin{aligned} I^0 &= -\frac{\mathcal{F}}{2}\mathcal{S}_{00} + \frac{-4a}{b\bar{\phi}}(-\mathcal{H}a\Lambda_{kk}) \\ &= -\frac{\mathcal{F}}{2}\mathcal{S}_{00} - \mathcal{H}\mathcal{S}_{kk}. \end{aligned} \tag{151}$$

Finally, plugging (151) in (142) gives

$$\begin{aligned} l^{0\nu}{}_{|\nu} &= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} - \mathcal{H}l\bar{g}^{00} + (2\mathcal{H} + \mathcal{B})\, l^{00} - \frac{1}{a^4} b \underbrace{\int \frac{1}{R}\left(\frac{\mathcal{F}'}{2}\mathcal{S}'_{00} + \mathcal{H}' f^{ik}\mathcal{S}'_{ik}\right) d^3x'}_{\text{treat } \mathcal{H}' \text{ and } \mathcal{F}' \text{ as constants}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} - \mathcal{H}l\bar{g}^{00} + \left(3\mathcal{H} - \frac{\mathcal{F}}{2}\right) l^{00} - \frac{\mathcal{F}}{2}Q^{00} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}Q_{ik} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} - \mathcal{H}l\bar{g}^{00} + \left(3\mathcal{H} - \frac{\mathcal{F}}{2}\right) l^{00} - \frac{\mathcal{F}}{2}\left(l^{00} + \bar{g}^{00}\frac{\varphi}{\bar{\phi}}\right) - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}\left(l_{ik} + \bar{g}_{ik}\frac{\varphi}{\bar{\phi}}\right) \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} - \mathcal{H}l\bar{g}^{00} + (3\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}\left(l_{ik} + \bar{g}_{ik}\frac{\varphi}{\bar{\phi}}\right) \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} - \mathcal{H}l\bar{g}^{00} + (3\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}l_{ik} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}\bar{g}_{ik}\frac{\varphi}{\bar{\phi}} \\
&\quad + \underbrace{\left(\mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00} - \mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00}\right)}_{\text{added and subtracted same thing}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} \underbrace{-\mathcal{H}l\bar{g}^{00}}_{1} + \underbrace{(3\mathcal{H} - \mathcal{F})\, l^{00}}_{2} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} \underbrace{-\mathcal{H}\frac{\bar{g}^{ik}}{a^2}l_{ik}}_{1} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}\bar{g}_{ik}\frac{\varphi}{\bar{\phi}} \\
&\quad + \left(\underbrace{\mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00}}_{1} - \underbrace{\mathcal{H}\bar{g}_{00}l^{00}\bar{g}^{00}}_{2}\right) \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} + (2\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \mathcal{H}\frac{\bar{g}^{ik}}{a^2}\bar{g}_{ik}\frac{\varphi}{\bar{\phi}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} + (2\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} + \mathcal{H}\bar{g}^{00}\bar{g}^{ik}\bar{g}_{ik}\frac{\varphi}{\bar{\phi}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} + (2\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} + 3\mathcal{H}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} + (2\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} + 3\mathcal{H}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} \\
&= \left(-\mathcal{H} + \frac{\mathcal{F}}{2}\right)\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}} + (2\mathcal{H} - \mathcal{F})\, l^{00} - \frac{\mathcal{F}}{2}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} + 3\mathcal{H}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} \\
&= (2\mathcal{H} - \mathcal{F})\, l^{00} + 2\mathcal{H}\bar{g}^{00}\frac{\varphi}{\bar{\phi}} - \frac{\varphi^{|0}}{\bar{\phi}}, \end{aligned} \tag{152}$$

in agreement with (86).

Now, for the i-th component,

$$\begin{aligned} I^i &= +\frac{\mathcal{F}}{2}\mathcal{S}_{i0} - \frac{-4a}{b\bar{\phi}}\left(-\mathcal{H}a\Lambda_{i0}\right) \\ &= +\frac{\mathcal{F}}{2}\mathcal{S}_{i0} + \mathcal{H}\mathcal{S}_{i0}, \end{aligned} \tag{153}$$

and thus,

$$\begin{aligned} l^{i\nu}{}_{|\nu} &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(2\mathcal{H}+\mathcal{B}\right)l^{i0} + \frac{1}{a^4}b\underbrace{\int\left(\mathcal{H}'+\frac{\mathcal{F}'}{2}\right)\frac{\mathcal{S}'_{i0}}{R}d^3x'}_{\text{treat }\mathcal{H}'\text{ and }\mathcal{F}'\text{ as constants}} \\ &\approx -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} + \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)\frac{1}{a^4}b\int\frac{\mathcal{S}'_{i0}}{R}d^3x' \\ &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} + \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)\frac{1}{a^4}l_{i0} \\ &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} + \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)\frac{1}{a^4}\bar{g}_{i\mu}\bar{g}_{0\nu}l^{\mu\nu} \\ &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} + \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)f_{i\mu}f_{0\nu}l^{\mu\nu} \\ &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} + \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)f_{ik}f_{00}l^{k0} \\ &= -\frac{\varphi^{|i}}{\bar{\phi}} + \left(3\mathcal{H}-\frac{\mathcal{F}}{2}\right)l^{i0} - \left(\mathcal{H}+\frac{\mathcal{F}}{2}\right)l^{i0} \\ &= \left(2\mathcal{H}-\mathcal{F}\right)l^{i0} - \frac{\varphi^{|i}}{\bar{\phi}}, \end{aligned} \tag{154}$$

as required.

VIII. APPENDIX: BACKGROUND FRIEDMAN COSMOLOGY

Here, for convenience, we list a few results related to the background Friedman cosmology.

A. General considerations

In this Subsection, the derivatives with respect to the coordinate time x^0, cosmic time T, conformal time η, and the scalar field ϕ will be denoted by

$$\frac{\partial F(x^0,x^i)}{\partial x^0} \equiv F_{,0}, \quad \frac{dF(T)}{dT} \equiv F_{,T}, \quad \frac{dF(\eta)}{d\eta} \equiv \dot{F}, \quad \frac{dF(\phi)}{d\phi} \equiv F'. \tag{155}$$

We work with the conformally *flat* background FLRW metric,

$$d\bar{g}^2 = a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right), \quad \bar{g}_{\alpha\beta} = a^2(\eta)f_{\alpha\beta}, \quad f_{\alpha\beta} = \mathrm{diag}\left(-1,1,1,1\right), \tag{156}$$

so that

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = -\frac{\mathcal{H}}{a}\left(\delta^{\alpha}_{\beta}\bar{u}_{\gamma} + \delta^{\alpha}_{\gamma}\bar{u}_{\beta} - \bar{u}^{\alpha}\bar{g}_{\beta\gamma}\right), \quad \bar{\Gamma}^0_{00} = \bar{\Gamma}^1_{10} = \bar{\Gamma}^2_{20} = \bar{\Gamma}^3_{30} = \bar{\Gamma}^0_{11} = \bar{\Gamma}^0_{22} = \bar{\Gamma}^0_{33} = \mathcal{H}, \tag{157}$$

$$\bar{R}_{\alpha\beta} = \frac{1}{a^2}\left[\dot{\mathcal{H}}\left(\bar{g}_{\alpha\beta} - 2\bar{u}_{\alpha}\bar{u}_{\beta}\right) + 2\mathcal{H}^2\left(\bar{g}_{\alpha\beta} + \bar{u}_{\alpha}\bar{u}_{\beta}\right)\right], \quad \bar{R}_{00} = -3\dot{\mathcal{H}}, \quad \bar{R}_{11} = \bar{R}_{22} = \bar{R}_{33} = \dot{\mathcal{H}} + 2\mathcal{H}^2, \tag{158}$$

$$\bar{R} = \frac{6}{a^2}\left(\dot{\mathcal{H}} + \mathcal{H}^2\right), \tag{159}$$

where the velocity of the Hubble flow is

$$\bar{u}^\mu = (1/a)\delta^\mu_0 = (1/a, 0, 0, 0), \quad \bar{u}_\mu = -a\delta^0_\mu = (-a, 0, 0, 0), \quad \bar{u}_\mu \bar{u}^\mu = -1, \tag{160}$$

and the (conformal) Hubble parameter is

$$\mathcal{H} \equiv \dot{a}/a. \tag{161}$$

Then, in the isotropic conformal coordinates,

$$\bar{\phi}_{|00} = \ddot{\bar{\phi}} - \mathcal{H}\dot{\bar{\phi}}, \quad \bar{\phi}_{|0i} = 0, \quad \bar{\phi}_{|ij} = -\mathcal{H}\dot{\bar{\phi}} f_{ij}, \tag{162}$$

$$\bar{\phi}^{|\alpha}_{|\alpha} = -\frac{1}{a^2}\left(\ddot{\bar{\phi}} + 2\mathcal{H}\dot{\bar{\phi}}\right), \tag{163}$$

$$\bar{\phi}^{|\alpha}\bar{\phi}_{|\alpha} = -\frac{1}{a^2}\dot{\bar{\phi}}^2, \tag{164}$$

and

$$\bar{T}^M_{\mu\nu} = (\bar{\epsilon} + \bar{p})\,\bar{u}_\mu \bar{u}_\nu + \bar{p}\bar{g}_{\mu\nu} = \mathrm{diag}(a^2\bar{\epsilon}, a^2\bar{p}, a^2\bar{p}, a^2\bar{p}), \tag{165}$$

$$\bar{T}^{\mu\nu}_M = (\bar{\epsilon} + \bar{p})\,\bar{u}^\mu \bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu} = \mathrm{diag}(\bar{\epsilon}/a^2, \bar{p}/a^2, \bar{p}/a^2, \bar{p}/a^2), \tag{166}$$

$$\bar{T}^M = \bar{g}^{\mu\nu}\bar{T}^M_{\mu\nu} = -\bar{\epsilon} + 3\bar{p}. \tag{167}$$

B. Background Friedman equations with conformal time

Combining the results of Subsection VIII A with Eqs. (21), (22), and (23), and calculating the sums

$$\bar{R}_{00} + \frac{1}{2}a^2\bar{R} - \frac{a^2}{\bar{\phi}}\bar{\phi}^{|\alpha}_{|\alpha}$$

and

$$\frac{1}{2}a^2\bar{R} - 3\mathcal{H}^2,$$

we get, in the isotropic conformal coordinates with conformal time η, the background Friedman equations,

$$3\mathcal{H}^2 = \frac{8\pi a^2}{\bar{\phi}}\bar{\epsilon} - 3\mathcal{H}\frac{\dot{\bar{\phi}}}{\bar{\phi}} + \frac{\omega}{2}\left(\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 + a^2\lambda, \tag{168}$$

$$3\dot{\mathcal{H}} = -\frac{8\pi a^2}{\bar{\phi}}\left[\frac{3+\omega}{3+2\omega}\bar{\epsilon} + \frac{\omega}{3+2\omega}(3\bar{p})\right] + 3\mathcal{H}\frac{\dot{\bar{\phi}}}{\bar{\phi}} - \left(\omega - \frac{3}{2}\frac{\omega'\bar{\phi}}{3+2\omega}\right)\left(\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 + \frac{a^2}{3+2\omega}\left(2\omega\lambda + 3\lambda'\bar{\phi}\right), \tag{169}$$

$$\ddot{\bar{\phi}} + 2\mathcal{H}\dot{\bar{\phi}} = \frac{8\pi a^2}{3+2\omega}(\bar{\epsilon} - 3\bar{p}) - \frac{\omega'}{3+2\omega}\dot{\bar{\phi}}^2. \tag{170}$$

C. Background Friedman equations with Hubble time

In this Subsection we work with the usual Hubble time, t, defined by

$$\frac{dt}{d\eta} \equiv a, \quad \mathcal{H} = Ha, \quad \frac{d\mathcal{H}}{d\eta} = \left(H^2 + \frac{dH}{dt}\right)a^2. \tag{171}$$

Then,

$$\frac{d^2\bar{\phi}}{d\eta^2} + 2\mathcal{H}\frac{d\bar{\phi}}{d\eta} = \left(\frac{d^2\bar{\phi}}{dt^2} + 3H\frac{d\bar{\phi}}{dt}\right)a^2. \tag{172}$$

For the remainder of this subsection the derivative with respect to t will be denoted with an overdot.

1. Friedman equations in general background cosmology

The background "matter" is assumed to be a fluid described by the equation of state,

$$\bar{p} = \alpha\bar{\epsilon}, \tag{173}$$

and obeying the law of conservation,

$$\bar{\epsilon}a^{3(1+\alpha)} = \bar{\epsilon}_0 a_0^{3(1+\alpha)}. \tag{174}$$

From (168), (169), (170), (171), (172), we get the background Friedmann equations,

$$\left(H + \frac{1}{2}\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 = \frac{3+2\omega}{12}\left(\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 + \frac{8\pi\bar{\epsilon}}{3\bar{\phi}} + \frac{\lambda}{3}, \tag{175}$$

$$H^2 + \dot{H} = -\frac{8\pi\bar{\epsilon}}{3\bar{\phi}}\left[\frac{3+(1+3\alpha)\omega}{3+2\omega}\right] + H\frac{\dot{\bar{\phi}}}{\bar{\phi}} - \left(\omega - \frac{3}{2}\frac{\omega'\bar{\phi}}{3+2\omega}\right)\left(\frac{\dot{\bar{\phi}}}{\bar{\phi}}\right)^2 + \frac{1}{3+2\omega}\left(2\omega\lambda + 3\lambda'\bar{\phi}\right), \tag{176}$$

$$\ddot{\bar{\phi}} + 3\dot{\bar{\phi}}H = \frac{8\pi}{3+2\omega}(1-3\alpha)\,\bar{\epsilon} - \frac{\omega'}{3+2\omega}\dot{\bar{\phi}}^2. \tag{177}$$

2. Friedman equations in standard Brans-Dicke background cosmology

The Brans-Dicke theory is recovered by setting

$$\lambda = 0, \quad \omega' = 0. \tag{178}$$

One possible solution maybe found in the power-law form by using the *Ansatz*,

$$a = a_0\left(\frac{t}{t_0}\right)^q, \quad \bar{\phi} = \bar{\phi}_0\left(\frac{t}{t_0}\right)^r, \quad \bar{\epsilon} = \bar{\epsilon}_0\left(\frac{t}{t_0}\right)^s, \tag{179}$$

so that

$$\dot{a} = a_0 q\frac{t^{q-1}}{t_0^q}, \quad \dot{\bar{\phi}} = \bar{\phi}_0 r\frac{t^{r-1}}{t_0^r}, \quad \ddot{\bar{\phi}} = \bar{\phi}_0 r(r-1)\frac{t^{r-2}}{t_0^r}, \quad H = \frac{1}{a}\frac{da}{dt} = \frac{q}{t}, \quad \frac{1}{\bar{\phi}}\frac{d\bar{\phi}}{dt} = \frac{r}{t}, \quad \frac{\bar{\epsilon}}{\bar{\phi}} = \frac{\bar{\epsilon}_0}{\bar{\phi}_0}\left(\frac{t}{t_0}\right)^{s-r}. \tag{180}$$

Using (178) and (180) in (174), (175) and (177), we get the system,

$$s + 3q(1+\alpha) = 0, \tag{181}$$

$$(2q+r)^2 = \left(1+\frac{2}{3}\omega\right)r^2 + \frac{4(3+2\omega)r(r-1+3q)}{3(1-3\alpha)}, \tag{182}$$

$$s - r + 2 = 0, \tag{183}$$

$$\bar{\phi}_0 = \frac{8\pi}{3+2\omega}\frac{(1-3\alpha)}{r(r-1+3q)}\bar{\epsilon}_0 t_0^2, \tag{184}$$

whose solution (which in the limit $\omega \to \infty$ correctly reproduces the standard Friedmann cosmology) is

$$q = \frac{2[1+(1-\alpha)\omega]}{4+3(1-\alpha^2)\omega}, \tag{185}$$

$$r = \frac{2(1-3\alpha)}{4+3(1-\alpha^2)\omega}, \tag{186}$$

$$s = \frac{6(1+\alpha)[1+(1-\alpha)\omega]}{4+3(1-\alpha^2)\omega}. \tag{187}$$

It is interesting to notice that

$$\frac{1}{H}\frac{\dot{\bar{\phi}}}{\bar{\phi}} = \frac{(1-3\alpha)}{1+(1-\alpha)\omega}, \tag{188}$$

and thus there are two small parameters in our theory,

$$\chi_1 = HT_0, \quad \chi_2 = \frac{\dot{\bar{\phi}}}{\bar{\phi}} T_0 = \frac{(1-3\alpha)}{1+(1-\alpha)\omega} HT_0, \tag{189}$$

where T_0 is the characteristic time of the dynamical evolution of the system, say, its orbital or rotational period. If

$$\alpha \neq 1, \quad \omega \gg 1, \tag{190}$$

which is a typical situation (currently accepted value is $\omega \simeq 4 \times 10^4$), then

$$\chi_1 \gg \chi_2. \tag{191}$$

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