Physics Letters A 382 (2018) 72–78

Contents lists available at ScienceDirect

Physics Letters A

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Ground state of an exciton in a three-dimensional parabolic quantum dot: Convergent perturbative calculation

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ARTICLE INFO

Article history:
Received 11 October 2017
Received in revised form 1 November 2017
Accepted 2 November 2017
Available online 3 November 2017
Communicated by V.A. Markel

Keywords:
Exciton
Quantum dot
Ground state
Convergent perturbation theory

ABSTRACT

Working in the effective-mass approximation, we apply a powerful convergent perturbative technique of Turbiner's to the calculation of the ground state energy and the wave function of an exciton confined to a three-dimensional parabolic quantum dot. Unlike the usual Rayleigh–Schrödinger perturbation theory, Turbiner's approach works well even in the regime of strong coupling and does not require the knowledge of the full solution to the undisturbed problem. The second-order convergent calculation presented below is in excellent agreement with the results of exact numerical simulations for a wide range of system's confinement parameters.

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1. Introduction

A quantum dot is a semiconductor nanocrystal consisting of 10^3 to 10^5 atoms and having a typical size of 2 to 10 nanometers [1]. The energy spectrum of a quantum dot is fundamentally different from that of a bulk semiconductor. A particle (such as an electron or a hole) in a nanocrystal behaves as if it was confined to a three-dimensional potential well, which makes particle's energy levels discrete and strongly dependent on the dot's size [2–4]. That leads to various quantum size effects which, in turn, give the dot its unique optoelectric properties [5,6].

Of particular interest to us is the influence of the confining potential on the ground state of an exciton [7,8] — an elementary excitation of the nanocrystal consisting of an electron and a hole interacting with each other by way of the usual electrostatic potential. Over the last thirty years, numerous approaches to the calculation of the ground state of the exciton have been employed by different authors. Chief among them are: the power series approach (also known as Frobenius method) [9–11], the standard Rayleigh–Schrödinger perturbation theory [12–14], the $1/N$ expansion [15,16], the expansion in electron-hole product states method [17], the variational method [18,14,19,20], the spin-density-functional-theory approach [21], the analytical iteration method with trial wave function [22], various numerical matrix diagonalization techniques [23,24,14,20], and the use of the finite-element numerical solvers [25]. Here we propose yet another approach — a powerful convergent perturbative method due to Turbiner [26–28], which works well even in the regime of strong coupling and does not require the knowledge of the full solution to the undisturbed problem.

In the effective-mass model, the exciton in a quantum dot is described by the Hamiltonian [29,30],

$$H_{\text{exciton}} = -\frac{\hbar^2}{2m_e^*}\nabla_e^2 - \frac{\hbar^2}{2m_h^*}\nabla_h^2 - \frac{e^2}{\epsilon|\mathbf{r}_e - \mathbf{r}_h|} + \frac{1}{2}m_e^*\omega^2 r_e^2 + \frac{1}{2}m_h^*\omega^2 r_h^2, \tag{1}$$

where $m_{e,h}^*$ and $\mathbf{r}_{e,h}$ are the (effective) masses and positions of the electron and the hole, $r_{e,h} \equiv |\mathbf{r}_{e,h}|$, e is the elementary charge, and ϵ is the dielectric constant of the ambient material. The last two terms in (1) represent the leading (harmonic) part of a more realistic confining potential. It is assumed that the frequency, ω, of small oscillations near the bottom of the well is independent of the particle's mass, which is similar to the case of a particle oscillating at the bottom of a frictionless bowl under the action of uniform gravity, familiar from introductory mechanics. The name "parabolic" used to describe the quantum dot comes from the shape of this confining potential [30].

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https://doi.org/10.1016/j.physleta.2017.11.001
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Introducing the center-of-mass and the relative coordinates [30,31],

$$\mathbf{R} = \frac{m_e^*\mathbf{r}_e + m_h^*\mathbf{r}_h}{m_e^* + m_h^*}, \quad \mathbf{r} = \mathbf{r}_e - \mathbf{r}_h, \tag{2}$$

we re-write system's Hamiltonian as the sum,

$$H_{\text{exciton}} = H_{\text{COM}} + H_{\text{relative}}, \tag{3}$$

where

$$H_{\text{COM}} = -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2 + \frac{M\omega^2 R^2}{2}, \tag{4}$$

$$H_{\text{relative}} = -\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2 - \frac{e^2}{\epsilon r} + \frac{m\omega^2 r^2}{2}, \tag{5}$$

and

$$M = m_e^* + m_h^*, \quad m = \frac{m_e^* m_h^*}{m_e^* + m_h^*}. \tag{6}$$

The system's dynamics, thus, separates into the center-of-mass motion of the exciton as a whole, which is described by H_{COM}, whose solution is trivial, and the motion of a fictitious particle of mass m described by H_{relative} with potential

$$V = -\frac{e^2}{\epsilon r} + \frac{m\omega^2 r^2}{2}. \tag{7}$$

Upon using some suitable units of length, r_0, and energy, $E_0 = \hbar^2/(2mr_0^2)$, the potential and the Schrödinger's equation for relative motion become

$$V = -\frac{2\alpha}{r} + br^2 \tag{8}$$

and

$$\nabla_{\mathbf{r}}^2\psi + (E - V)\psi = 0, \tag{9}$$

respectively, with dimensionless r, E, and constants

$$\alpha \equiv \frac{me^2}{\epsilon\hbar^2}r_0, \quad b \equiv \frac{m^2\omega^2}{\hbar^2}r_0^4. \tag{10}$$

Our goal will be to find the ground state of H_{relative} for *any* values of α and b.

[Notice that the possibility of having analytical solutions for some particular, denumerably infinite set of oscillator frequencies was first pointed out in Ref. [32]. The fact that those solutions are a reflection of the hidden $sl(2)$ algebraic structure and, hence, the problem is quasi-exactly solvable and is equivalent to $sl(2)$ quantum top in a magnetic field, was emphasized in Ref. [33]. It was then shown in Refs. [34–36] that the problem is, in fact, equivalent to the problem of two Coulomb charges on a plane subjected to a constant magnetic field orthogonal to that plane.]

2. Formal perturbative calculation

To determine the ground state of the exciton, we use the convergent perturbation theory developed by Turbiner [26–28]. Turbiner's theory consists of two parts. The first part (used in this section) deals with the formal development of the perturbative scheme applicable to any potential, V, that can be split into a formal sum,

$$V = V_0 + \lambda V_1, \tag{11}$$

of its zeroth-order (unperturbed) component, V_0, and the perturbation, λV_1, where λ is the perturbation parameter. At this stage, no specific requirements are imposed on the perturbation other than that it has to be small in some reasonable sense. For example, in the case of a quantum dot, we may choose

$$r_0 = \frac{\epsilon\hbar^2}{me^2}, \quad \alpha = 1, \quad b = \frac{\omega^2\hbar^6\epsilon^4}{m^2e^8}, \tag{12}$$

and split the potential in (8) according to

$$V_0 = -\frac{2}{r}, \quad \lambda V_1 = br^2, \tag{13}$$

which would correspond to the so-called weak-confinement regime [30], with b playing the role of the perturbation parameter. Alternatively, we could choose

$$r_0 = \sqrt{\frac{\hbar}{m\omega}}, \quad b = 1, \quad \alpha = \frac{e^2}{\epsilon}\sqrt{\frac{m}{\omega\hbar^3}}, \tag{14}$$

and

$$V_0 = r^2, \quad \lambda V_1 = -\frac{2\alpha}{r}, \tag{15}$$

and work in the strong-confinement regime [30], with small parameter α. Regardless of the V-split, the ground state wave function, ψ (here assumed to be nodeless; for the possibility of having a nodal ground state see [37]), is sought in the form (here, e is the base of the natural logarithm)

$$\psi(\mathbf{r}) = e^{-\phi(\mathbf{r})}, \tag{16}$$

with ϕ being a power series in λ,

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \ldots, \tag{17}$$

and, similarly, for the ground state energy,

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 + \ldots. \tag{18}$$

The actual perturbative calculation is not applied to ϕ directly, however. Instead, Turbiner introduces an auxiliary vector field, $\mathbf{y}$, defined by

$$\mathbf{y} = -\nabla\psi/\psi = \nabla\phi, \tag{19}$$

which satisfies the non-linear differential equation (hence the name "nonlinearization procedure" used in his papers),

$$\nabla\cdot\mathbf{y} - \mathbf{y}^2 = E - V, \tag{20}$$

with the boundary condition,

$$|\psi^2\mathbf{y}| \to 0, \quad r\to\infty, \tag{21}$$

which corresponds to the requirement that the probability current at infinity must vanish. The series expansion for $\mathbf{y}$,

$$\mathbf{y} = \mathbf{y}_0 + \lambda\mathbf{y}_1 + \lambda^2\mathbf{y}_2 + \ldots, \tag{22}$$

is then found using the iteration scheme,

$$\nabla\cdot\mathbf{y}_n - 2\mathbf{y}_0\cdot\mathbf{y}_n = E_n - Q_n, \quad n = 1,2,3,\ldots, \tag{23}$$

where

$$\mathbf{y}_0 = -\frac{\nabla\psi_0}{\psi_0}, \tag{24}$$

$$Q_1 = V_1, \tag{25}$$

$$E_n = \frac{\int \psi_0^2\, Q_n\, dV}{\int \psi_0^2\, dV}, \quad n = 1,2,3,\ldots, \tag{26}$$

$$Q_n = -\sum_{i=1}^{n-1}\mathbf{y}_i\cdot\mathbf{y}_{n-i}, \quad n = 2,3,\ldots, \tag{27}$$

with dV being the volume element in the configuration space of the system. The remarkable feature of Turbiner's method, which makes it very different from the usual perturbation theory, is that it does not require the complete solution (all wave functions and the full spectrum) of the undisturbed problem.

In the spherically-symmetric case, all $\mathbf{y}_n$ have the form,

$$\mathbf{y}_n = y_n(r)\hat{\mathbf{r}}, \quad n = 0,1,2,3,\ldots, \tag{28}$$

with $\hat{\mathbf{r}} = \mathbf{r}/r$, and (23) reduces to a much simpler sequence of first order linear ordinary differential equations,

$$ry_n' = 2(y_0 r - 1)y_n + r(E_n - Q_n), \quad n = 1,2,3,\ldots, \tag{29}$$

each of which has the general solution,

$$y_n(r) = C_n e^{F(r)} + e^{F(r)}\int e^{-F(r)}[E_n - Q_n(r)]\,dr, \tag{30}$$

with

$$F(r) = \int \frac{2(y_0 r - 1)}{r}\,dr, \tag{31}$$

where the constants C_n must be determined from the corresponding boundary conditions,

$$|\psi_0^2 y_n| \to 0, \quad r\to\infty. \tag{32}$$

Applying this scheme to the case of a weakly confined dot, Eq. (13), we first choose

$$\psi_0 = e^{-r}, \quad E_0 = -1, \tag{33}$$

and then find, upon direct calculation,

$$E = -1 + 3b - \frac{129}{8}b^2 + \frac{5451}{16}b^3 - \frac{6609975}{512}b^4 + \ldots, \tag{34}$$

$$\phi = r + \frac{br^2}{6}(3+r) - \frac{b^2r^2}{480}(1290 + 430r + 105r^2 + 12r^3) + \frac{b^3r^2}{6720}(381570 + 127190r + 34545r^2 + 6804r^3 + 910r^4 + 60r^5) + \ldots. \tag{35}$$

Similarly, in the strongly confined case, Eq. (15), we set

$$\psi_0 = e^{-r^2/2}, \quad E_0 = 3, \tag{36}$$

and get

$$E = 3 - \frac{4\alpha}{\sqrt{\pi}} - \frac{4\alpha^2}{\sqrt{\pi}}\int_0^\infty e^{-r^2}\left\{\frac{1}{r}\left[e^{r^2}\operatorname{erfc}(r) - 1\right] + \frac{2}{\sqrt{\pi}}\right\}^2 dr + \ldots, \tag{37}$$

$$\phi = \frac{r^2}{2} + \alpha\int_0^r \left\{\frac{1}{\tilde{r}^2}\left[e^{\tilde{r}^2}\operatorname{erfc}(\tilde{r}) - 1\right] + \frac{2}{\sqrt{\pi}}\frac{1}{\tilde{r}}\right\} d\tilde{r} + \ldots, \tag{38}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \sim \frac{2e^{-x^2}}{\sqrt{\pi}}\left(x + \frac{2x^3}{1\cdot 3} + \frac{4x^5}{1\cdot 3\cdot 5} + \ldots\right), \quad x \to 0, \tag{39}$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{3}{4x^5} - \ldots\right), \quad x \to \infty, \tag{40}$$

are the error and complementary error functions.

The main problem with the formal perturbative scheme (be it the usual one, or the one described above) is evident from Eqs. (34), (35) describing the weakly confined quantum dot. The resulting series for the ground state energy and for the exponent of the wave function are not actually convergent, but asymptotic. As with any asymptotic expansion, one has to exercise care when interpreting the calculated results. The asymptotic expansion, by its very nature, works best in the limit when perturbation approaches zero, in which case one may keep many terms in the series. However, when the perturbation increases, the expansion begins to break down and only a small number of terms may be kept, thus reducing the accuracy of the final result. One describes this situation formally by saying that the series has zero radius of convergence. This is very different from a series whose radius of convergence is different from zero: in that case, the more terms one keeps, the better the resulting approximation.

3. Convergent approach

The reason for the appearance of diverging series (34) and (35) in our calculation is that, for sufficiently large r, the perturbation potential, $V_1 = br^2$, becomes larger than the unperturbed potential, $V_0 = -2\alpha/r$. Because of that, V_1 cannot be regarded as a "good" perturbation acting on the unperturbed system.

To develop a perturbative scheme that is *convergent* (and works even for strong perturbations), Turbiner formulates the following prescription: choose V_0 in such a way that it reproduces as many of the characteristic properties of the full potential V as possible, such as, e.g., its singular points, asymptotic behavior, etc.

To see how this prescription works in the case of a quantum dot subjected to an arbitrary parabolic potential (8), we first choose

$$\psi_0 = e^{-\alpha r - \sqrt{b}r^2/2} \tag{41}$$

as the zeroth-order approximation to the ground state wave function. This is a very reasonable choice, for in both critical limits, $r \to 0$ and $r \to \infty$, the behavior of ψ_0 matches the behavior of the ground state wave function of the corresponding dominating potential, $-2\alpha/r$ or br^2, respectively. [Notice that (41) is similar in form to the "variational wavefunction" proposed in Ref. [19], though in that case α and b were treated as the variational parameters.] The corresponding zeroth order potential, V_0, and the ground state energy, E_0, are then readily found by calculating the quantity

$$\frac{\nabla^2\psi_0}{\psi_0} = \left(-\frac{2\alpha}{r} + br^2 + 2\alpha\sqrt{b}r\right) - \left(-\alpha^2 + 3\sqrt{b}\right), \tag{42}$$

which, on the basis of the zeroth order Schrödinger equation, should be identified with the difference $V_0 - E_0$. From (42), we immediately get

$$V_0 = -\frac{2\alpha}{r} + br^2 + 2\alpha\sqrt{b}r, \tag{43}$$

$$E_0 = -\alpha^2 + 3\sqrt{b}, \tag{44}$$

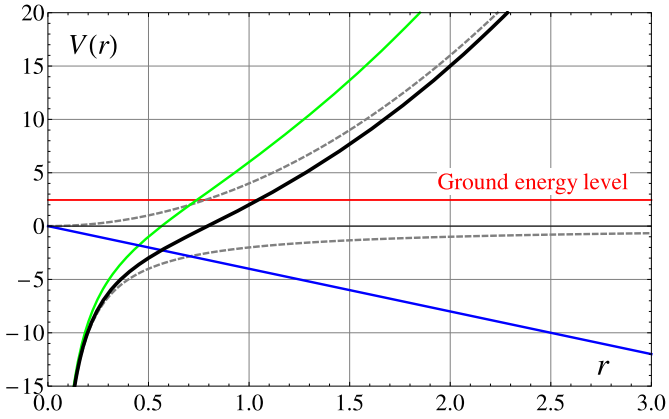

Fig. 1. (Color online.) Black curve: Graph of excitonic potential energy, $V(r)$, Eq. (8), at $\alpha = 1$ and $b = 4$. Gray dashed curves: the Coulomb and harmonic oscillator potentials comprising V. Green curve: zeroth-order potential, V_0, as given in Eq. (43). Blue curve: "good" perturbation, V_1, as given in Eq. (45). Red curve: ground level, with energy $E_{\text{num}} = 2.4474$ (numerically calculated value). Second-order convergent perturbative calculation gives $E = 2.4632$; see below.

as well as the "good" perturbation (see Fig. 1),

$$\lambda V_1 = -2\alpha\sqrt{b}r. \tag{45}$$

The asymptotic behavior of the found V_0 correctly reproduces the behavior of the full potential V at its critical points, as required by Turbiner's prescription. Additionally, the perturbation V_1 never dominates the zeroth-order V_0 for any value of r.

From here on, the calculation proceeds in accordance with the general scheme outlined in Section 2. Using (24), (25), (26), (27), and (30), we find

$$y_1 = \frac{1}{r^2}\left[C_1 e^{r\left(2\alpha+\sqrt{b}r\right)} - \frac{\alpha\left(\sqrt{\pi}e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)\left(2\alpha^2r^2+2\alpha r+\sqrt{b}r^2+1\right)+\sqrt{\pi}e^{\frac{\left(\alpha+\sqrt{b}r\right)^2}{\sqrt{b}}}\operatorname{erf}\left(\frac{\alpha+\sqrt{b}r}{\sqrt[4]{b}}\right)-2\sqrt[4]{b}r(\alpha r+1)\right)}{\sqrt{\pi}\left(2\alpha^2+\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\alpha\sqrt[4]{b}}\right], \tag{46}$$

and, upon setting

$$C_1 = \frac{\sqrt{\pi}\alpha e^{\frac{\alpha^2}{\sqrt{b}}}}{\sqrt{\pi}\left(2\alpha^2+\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\alpha\sqrt[4]{b}}, \tag{47}$$

get

$$y_1 = -\frac{\alpha\left[\sqrt{\pi}e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)\left(2\alpha^2r^2+2\alpha r+\sqrt{b}r^2+1\right)-\sqrt{\pi}e^{\frac{\left(\alpha+\sqrt{b}r\right)^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha+\sqrt{b}r}{\sqrt[4]{b}}\right)-2\sqrt[4]{b}r(\alpha r+1)\right]}{r^2\left[\sqrt{\pi}\left(2\alpha^2+\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\alpha\sqrt[4]{b}\right]}, \tag{48}$$

which vanishes at the origin and satisfies the boundary condition, (32), at infinity. We then get the first-order correction to the ground state energy,

$$E_1 = \frac{2\alpha\left[\sqrt{\pi}\alpha\left(2\alpha^2+3\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\sqrt[4]{b}\left(\alpha^2+\sqrt{b}\right)\right]}{\sqrt{\pi}\left(2\alpha^2+\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\alpha\sqrt[4]{b}}, \tag{49}$$

the second-order correction,

$$E_2 = \frac{-4\alpha^2b^{5/4}}{\left[\sqrt{\pi}\left(2\alpha^2+\sqrt{b}\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)-2\alpha\sqrt[4]{b}\right]^3}\int_0^\infty \frac{e^{-r\left(2\alpha+\sqrt{b}r\right)}}{r^2}\left[\sqrt{\pi}\left(2\alpha^2r^2+2\alpha r+\sqrt{b}r^2+1\right)e^{\frac{\alpha^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)\right.$$
$$\left.-\sqrt{\pi}e^{\frac{\left(\alpha+\sqrt{b}r\right)^2}{\sqrt{b}}}\operatorname{erfc}\left(\frac{\alpha+\sqrt{b}r}{\sqrt[4]{b}}\right)-2\sqrt[4]{b}r(\alpha r+1)\right]^2 dr, \tag{50}$$

and the exponent of the wave function in first order,

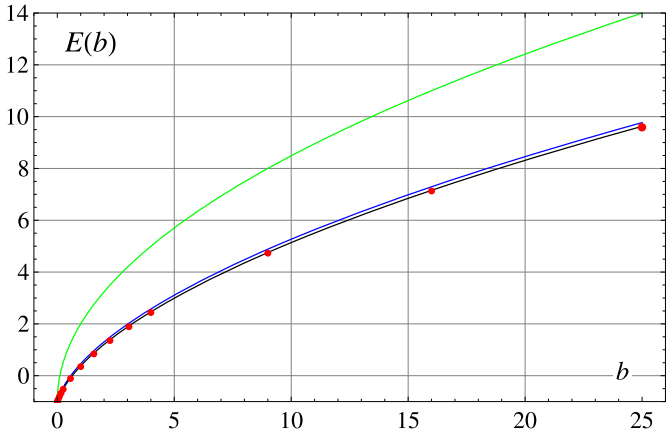

Fig. 2. (Color online.) Ground state energy, E, of the exciton, Eq. (8), at $\alpha = 1$ as a function of the parabolicity parameter, b. Green curve: plot of zeroth-order approximation, $E \approx E_0$, using Eq. (44). Blue curve: plot of the first-order approximation, $E \approx E_0 + E_1$, using Eqs. (44) and (49). Black curve: plot of the second-order approximation, $E \approx E_0 + E_1 + E_2$, using Eqs. (44), (49), and (50). Red dots: exact numerical simulation. For example, at $b = 4$, we have $E_0 = 5$, $E_1 = -2.42345$, $E_2 = -0.113349$, $E = 2.4632$, and $E_{\text{num}} = 2.4474$.

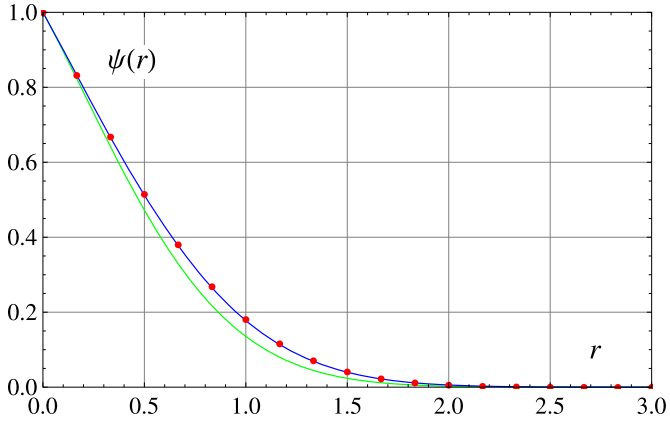

Fig. 3. (Color online.) Ground state wave function, $\psi(r)$, of the exciton, Eq. (8), at $\alpha = 1$ and $b = 4$. Green curve: plot of the zeroth-order approximation, ψ_0, using Eq. (41). Blue curve: plot of the first-order approximation, $\psi = e^{-\phi(r)}$, with ϕ given in Eq. (51). Red dots: exact numerical simulation.

$$\phi = \alpha r + \frac{\sqrt{b}}{2} r^2 + \frac{-\alpha}{\sqrt{\pi}\left(2\alpha^2 + \sqrt{b}\right) e^{\frac{\alpha^2}{\sqrt{b}}} \operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right) - 2\alpha\sqrt[4]{b}}$$
$$\times \int_0^r \frac{\sqrt{\pi} e^{\frac{\alpha^2}{\sqrt{b}}} \operatorname{erfc}\left(\frac{\alpha}{\sqrt[4]{b}}\right)\left(2\alpha^2\tilde{r}^2 + 2\alpha\tilde{r} + \sqrt{b}\tilde{r}^2 + 1\right) - \sqrt{\pi} e^{\frac{\left(\alpha+\sqrt{b}\tilde{r}\right)^2}{\sqrt{b}}} \operatorname{erfc}\left(\frac{\alpha+\sqrt{b}\tilde{r}}{\sqrt[4]{b}}\right) - 2\sqrt[4]{b}\tilde{r}(\alpha\tilde{r}+1)}{\tilde{r}^2} d\tilde{r}, \tag{51}$$

with the corresponding results depicted in Figs. 2 and 3. As seen from those figures, Turbiner's method works exceptionally well for a wide range of confinement parameters.

Notice that the found E_1 coincides with the first-order correction to the ground state energy calculated on the basis of the usual perturbation theory,

$$E_1 = \frac{\langle\psi_0|V_1|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} \equiv \frac{\int_0^\infty V_1 \psi_0^2 r^2 dr}{\int_0^\infty \psi_0^2 r^2 dr}. \tag{52}$$

Notice also that the second-order correction,

$$E_2 = -\frac{\int_0^\infty y_1^2 \psi_0^2 r^2 dr}{\int_0^\infty \psi_0^2 r^2 dr}, \tag{53}$$

is always negative, as should have been expected. These two general properties of Turbiner's scheme had already been pointed out in the original papers on the subject [26–28].

Additionally, for the purposes of engineering applications, given the value of the ground-state energy, E_{given}, one may want to calculate the required value of the confinement parameter, b_{req}. This can be easily achieved with the help of the Nelder–Mead [38] simplex direct search applied to the approximate equation,

$$E_{\text{given}} \approx E_0 + E_1 + E_2, \tag{54}$$

with E_0, E_1, and E_2 given in Eqs. (44), (49), and (50). For example, for $\alpha = 1$ and $E_{\text{given}} = 0$, we find $b_{\text{req}} \approx 0.6370$, with the actual value being $b \approx 0.6507$.

4. Summary

Assuming the effective mass model, we performed a convergent perturbative calculation of the ground state of the exciton confined by the parabolic potential of a three-dimensional quantum dot. No use of the full solution of the unperturbed problem has been made, except for the easily determined expressions for the unperturbed ground state energy and the wave function. For a wide range of system's parameters, our approximate solution is in very good agreement with the results of exact numerical simulations.

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