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RESEARCH ARTICLE

Neutron interference in the Earth's gravitational field

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Abstract This work relates to the famous experiments, performed in 1975 and 1979 by Werner et al., measuring neutron interference and neutron Sagnac effects in the earth's gravitational field. Employing the method of Stodolsky in its weak field approximation, explicit expressions are derived for the two phase shifts, which turn out to be in agreement with the experiments and with the previously obtained expressions derived from semi-classical arguments: these expressions are simply modified by relativistic correction factors.

Keywords Neutron interference · Kerr metric · COW effect · Sagnac effect

1 Introduction

It is now several decades since the ground-breaking work by Werner and his co-workers showed that gravitational [1,2] and rotational [3] effects were to be found in neutron interference experiments performed on the earth's surface [4–6]. The predicted and experimentally confirmed gravitational phase shift is the only expression in physics to feature both Newton's constant of gravitation G and Planck's quantum of action $\hbar$, which surely makes these experiments particularly noteworthy. The two experiments are referred to hereafter as the COW experiment and the neutron Sagnac effect.

Straightforward, semi-classical derivations of these effects have already appeared in the literature (see for example [7–9]) and in abbreviated form are summarized in Sects. 2 and 3 below. What is very clear, however, is that a proper account of this topic

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should really be sought in General Relativity (GR)—which is, after all, a theory of gravity!—and indeed numerous papers have been written using this approach (see for example [10–15]). Some of these explore rather sophisticated notions, for example a possible parallel between the COW experiment and the Aharonov–Bohm effect, based on the integrated curvature of an enclosed path, on the one hand in parameter space and on the other hand in field space [14]. We do not aim to explore these higher-flown topics, but rather to present a simple demonstration of how GR can account for the findings in neutron interferometry and to show that it has an application in quantum physics [16]—a notion which might still cause some surprise!

We use the Kerr solution of GR [17,18], since this includes the rotation of the earth through the angular momentum parameter a, as well as ω, the angular velocity of the earth, and r_s its Schwarzschild radius. These are all small parameters, and we calculate the relevant effects to second order in all these quantities (mixed and unmixed). The general method of procedure is the weak field approximation, adopted by Stodolsky [11] (also Linet and Tourrenc [10]), together with the use of the geodesic equation, or the eikonal method of geometrical optics.

The next Section describes a standard, elementary derivation of the COW effect, and in Sect. 3 is a similarly elementary derivation of the Sagnac effect for neutrons. In Sect. 4 the general relativistic setting for more realistic derivations of these effects is presented. The Kerr metric is displayed as well as a coordinate transformation to a Cartesian system relevant to our problem. In the final Section our results are derived. Use is made of the weak field approximation in conjunction with a specific assumption which allows the calculations to be preformed. It is found that the resulting phase shifts are, in both cases, those predicted by the simple models in Sects. 2 and 3, with correction factors of $\gamma = (1 - v^2/c^2)^{-1/2}$, and additional small terms involving ω, the angular velocity of the earth.

2 Simple derivation of COW effect

The setup described in reference [2] (see also [9]) is based on the splitting of the neutron beam by Bragg diffraction from perfect crystals, as first implemented for X rays by Bonse and Hart [19]. Rauch and Werner [9] point out that when the desired degree of crystal cutting is achieved, the resulting interferometry "exhibits the fundamentals of quantum mechanics in a very direct and obvious way". The interference involved is "topologically equivalent to a ring", which we represent as a rectangle, of macroscopic dimensions (centimetres). The neutrons enter at the bottom left corner where the beam splits into two, and the beams recombine at the top right corner, where the interference takes place.

The spatial part of a plane matter wave describing a neutron beam is given by $e^{i\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k}$ is the wave vector and $k \equiv |\mathbf{k}| = 2\pi/\lambda$ is the wave number, with λ being the de Broglie wavelength, so the phase accumulated over a path from $\mathbf{r}_0$ to $\mathbf{r}$ is

$$\Phi(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{k} \cdot d\mathbf{r}, \tag{1}$$

or, since $\lambda/(2\pi) = \hbar/p$, where p is particle's momentum,

$$\Phi(\mathbf{r}) = \frac{1}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{p}\cdot d\mathbf{r}. \tag{2}$$

This refers to a particular *path*, so the phase difference between neutron beams along two distinct paths is

$$\Delta\Phi = \frac{1}{\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} (\mathbf{p}_{\mathrm{I}} - \mathbf{p}_{\mathrm{II}})\cdot d\mathbf{r}. \tag{3}$$

In our case the path I is the lower route and path II the upper route. The contributions to $\Delta\Phi$ from the vertical parts of these two routes cancel, since the relevant momenta are equal and opposite; and putting $p_{\mathrm{I}} = mv$ and $p_{\mathrm{II}} = mu$ along the horizontal lower and upper routes respectively (with v and u being the corresponding particle speeds), we find

$$\Delta\Phi = \frac{1}{\hbar} m(v-u)L, \tag{4}$$

where L is the length of the interferometer. Conservation of energy now gives us

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 - mgH \tag{5}$$

where g is the acceleration due to gravity and H the height of the interferometer. Since gH is of the order of 10^{-1} $\mathrm{m^2 s^{-2}}$ and $v^2 \approx 4\times 10^6\,\mathrm{m^2 s^{-2}}$ for thermal neutrons, then $gH \ll v^2$, and

$$v - u \approx \frac{gH}{v}, \tag{6}$$

giving finally

$$\Delta\Phi = \frac{mgA}{\hbar v}, \tag{7}$$

where $A = LH$ is the area of the interferometer. This phase shift was first predicted and observed in 1975 by Colella, Overhauser and Werner [2].

It is pertinent to note that the above expression for the phase shift may alternatively be obtained by starting from a Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \frac{p^2}{2m} + m\mathbf{g}\cdot\mathbf{r}, \tag{8}$$

with $\mathbf{p}$ $(= m\mathbf{v} = m\dot{\mathbf{r}})$ defined by

$$\mathbf{p} = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}. \tag{9}$$

Equation (5) then yields the expected result (7).

3 Simple derivation of the neutron Sagnac effect

The experiment, first performed by Werner, Staudenmann and Colella [3], measured the effect of the earth's rotation on the neutron phase. To take account of a rotating frame the Lagrangian (8) should be modified to

$$\mathcal{L} = \frac{p^2}{2m} + m\mathbf{g}\cdot\mathbf{r} + \boldsymbol{\omega}\cdot\boldsymbol{\ell}, \tag{10}$$

where $\boldsymbol{\omega}$ is the Earth's angular velocity, with magnitude $\omega \equiv |\boldsymbol{\omega}|$, and $\boldsymbol{\ell}$ is the angular momentum of the particle. Then the momentum (9) becomes

$$\mathbf{p} = m\mathbf{v} + m\boldsymbol{\omega}\times\mathbf{r}. \tag{11}$$

The phase coming from the term in ω is

$$\Delta\alpha = \frac{1}{\hbar}\oint m[\boldsymbol{\omega}\times\mathbf{r}]\cdot d\mathbf{r} = \frac{2m\boldsymbol{\omega}\cdot\boldsymbol{A}}{\hbar}, \tag{12}$$

where $\boldsymbol{A}$ is the vector orthogonal to the area enclosed by the path followed by the particle, with magnitude $A \equiv |\boldsymbol{A}|$. This Sagnac phase is typically of the order of 10^{-2} of the gravitational COW phase, so to detect it means setting up the apparatus in such a way that the COW contribution to the phase is zero. This is achieved by having the interferometer in a vertical plane—say in the $r\theta$ or $r\phi$ plane—and then rotating it about a vertical axis. The observations of the phase shift of the neutron due to the earth's rotation were found to be in good agreement with the theory [3].

4 Kerr metric and the Earth's gravitational field

We now turn to a general relativistic derivation of the COW and neutron Sagnac effects. To describe the gravitational field of the rotating earth, we first write down the Kerr metric [17] in its standard Boyer–Lindquist form [18],

$$\begin{aligned} ds^2 = &\left(1 - \frac{r_s}{r}\frac{1}{1+\left(\frac{a}{r}\right)^2\cos^2\theta}\right)c^2dt^2 + 2\frac{r_s}{r}\frac{a}{r}\frac{\sin^2\theta}{1+\left(\frac{a}{r}\right)^2\cos^2\theta}r(cdt)d\varphi \\ &- \frac{1+\left(\frac{a}{r}\right)^2\cos^2\theta}{1-\frac{r_s}{r}+\left(\frac{a}{r}\right)^2}dr^2 - \left(1+\left(\frac{a}{r}\right)^2\cos^2\theta\right)r^2d\theta^2 \\ &- \left(1+\left(\frac{a}{r}\right)^2 + \frac{r_s}{r}\left(\frac{a}{r}\right)^2\frac{\sin^2\theta}{1+\left(\frac{a}{r}\right)^2\cos^2\theta}\right)r^2\sin^2\theta d\varphi^2, \end{aligned} \tag{13}$$

where $a = (2/5)R^2\omega/c$ is the angular momentum parameter, $r_s = 2GM/c^2$ is the Schwarzschild radius, M is the mass, and R is the radius of the earth. This metric describes the gravitational field of the rotating earth as seen from an inertial frame. The experiments we are considering, however, take place on the earth's surface, and therefore in a rotating frame, so to find the appropriate metric we must replace φ by φ' given by

$$\varphi = \varphi' + \omega t, \tag{14}$$

in which the metric becomes

$$
\begin{aligned}
ds^2 = & \left\{ 1 - \frac{r_s}{r} \frac{1}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} + 2 \frac{r_s}{r} \frac{a}{r} \frac{r\omega}{c} \frac{\sin^2\theta}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} \right. \\
& \left. - \frac{r^2\omega^2}{c^2} \left[1 + \left(\frac{a}{r}\right)^2 + \frac{r_s}{r} \left(\frac{a}{r}\right)^2 \frac{\sin^2\theta}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} \right] \sin^2\theta \right\} c^2 dt^2 \\
& + 2 \left\{ \frac{r_s}{r} \frac{a}{r} \frac{\sin^2\theta}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} \right. \\
& \left. - \frac{r\omega}{c} \left[1 + \left(\frac{a}{r}\right)^2 + \frac{r_s}{r} \left(\frac{a}{r}\right)^2 \frac{\sin^2\theta}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} \right] \sin^2\theta \right\} r(cdt) d\varphi' \\
& - \frac{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta}{1 - \frac{r_s}{r} + \left(\frac{a}{r}\right)^2} dr^2 - \left[1 + \left(\frac{a}{r}\right)^2 \cos^2\theta \right] r^2 d\theta^2 \\
& - \left[1 + \left(\frac{a}{r}\right)^2 + \frac{r_s}{r} \left(\frac{a}{r}\right)^2 \frac{\sin^2\theta}{1 + \left(\frac{a}{r}\right)^2 \cos^2\theta} \right] r^2 \sin^2\theta d\varphi'^2 .
\end{aligned} \tag{15}
$$

This expression is exact. Taking into account that for $r \approx R$ (radius of the earth), $r_s/r \sim 10^{-9}$, $\omega r/c \sim 10^{-6}$, $a/r \sim 10^{-6}$, $a\omega/c \sim 10^{-12}$, we expand to order 10^{-15} and get

$$
\begin{aligned}
ds^2 = & \left(1 - \frac{r_s}{r} - \frac{r^2\omega^2}{c^2} \sin^2\theta \right) c^2 dt^2 \\
& + 2 \left(\frac{r_s}{r} \frac{a}{r} - \frac{r\omega}{c} \right) r \sin^2\theta d\varphi' (cdt) \\
& - \left[1 + \frac{r_s}{r} - \left(\frac{a}{r}\right)^2 \sin^2\theta \right] dr^2 \\
& - \left[1 + \left(\frac{a}{r}\right)^2 \cos^2\theta \right] r^2 d\theta^2 - \left[1 + \left(\frac{a}{r}\right)^2 \right] r^2 \sin^2\theta d\varphi'^2 .
\end{aligned} \tag{16}
$$

It is convenient to rewrite (16) in terms of the "shifted" Cartesian coordinates erected on the surface of the earth, by analogy with how it was done in the Schwarzschild case in Ref. [20]. The idea is to work in a coordinate system whose origin is "in the laboratory", on the earth's surface, and also that this should be a Cartesian system, since this simplifies the calculation.

We first introduce the "usual" Cartesian coordinates (x, y, z) defined by

$$
r = \left(x^2 + y^2 + z^2 \right)^{1/2}, \quad \varphi' = \arctan \frac{y}{x}, \quad \theta = \arccos \frac{z}{r}, \tag{17}
$$

and get

$$
\begin{aligned}
ds^2 = & \left[1 - \frac{r_s}{r} - \frac{\omega^2}{c^2} (x^2 + y^2) \right] c^2 dt^2 \\
& + 2 \left(\frac{r_s}{r} \frac{a}{r} - \frac{r\omega}{c} \right) \frac{x dy - y dx}{r} (cdt)
\end{aligned}
$$

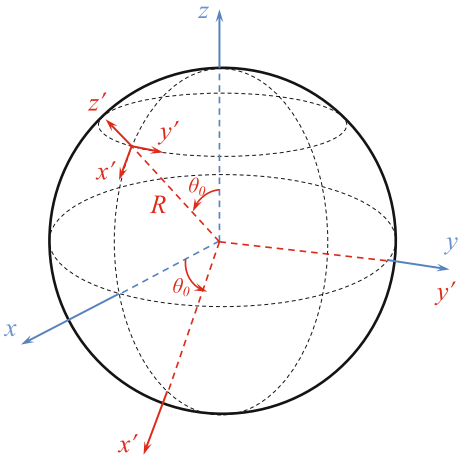

Fig. 1 (Color online) Local Cartesian coordinates on the surface of the rotating Earth

$$
\begin{aligned}
&- \left[1 + \frac{r_s}{r} - \left(\frac{a}{r}\right)^2 \frac{x^2+y^2}{r^2}\right] \frac{(xdx + ydy + zdz)^2}{r^2} \\
&- \left[1 + \left(\frac{a}{r}\right)^2 \frac{z^2}{r^2}\right] \frac{\left(zxdx + zydy - (x^2+y^2)dz\right)^2}{r^2(x^2+y^2)} \\
&- \left[1 + \left(\frac{a}{r}\right)^2\right] \frac{(xdy - ydx)^2}{x^2+y^2}. \qquad (18)
\end{aligned}
$$

We next perform the rotation around the y-axis by an angle θ_0 (the co-latitude of interferometer location on the earth's surface; see Fig. 1) and shift the origin by R along the new z-axis in accordance with

$$x = x' \cos\theta_0 + (R + z') \sin\theta_0, \qquad (19)$$
$$y = y', \qquad (20)$$
$$z = -x' \sin\theta_0 + (R + z') \cos\theta_0, \qquad (21)$$

and

$$dx = dx' \cos\theta_0 + dz' \sin\theta_0, \qquad (22)$$
$$dy = dy', \qquad (23)$$
$$dz = -dx' \sin\theta_0 + dz' \cos\theta_0, \qquad (24)$$

where x', y' and z' are the Cartesian coordinates whose origin is on the earth's surface. We now restrict the experimental region to the neighborhood of this shifted origin and introduce the weak field approximation, in which $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $|h_{\mu\nu}| \ll 1$, with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We finally obtain, to terms linear in x'/R, y'/R and z'/R,

$$h_{00} = -\frac{r_s}{R}\left(1-\frac{z'}{R}\right) - \frac{\omega^2 R^2}{c^2}\left[\left(1+\frac{2z'}{R}\right)\sin^2\theta_0 + \frac{x'}{R}\sin(2\theta_0)\right], \quad (25)$$

$$h_{01} = \left(\frac{\omega R}{c} - \frac{a r_s}{R^2}\right)\frac{y'}{R}\cos\theta_0, \quad (26)$$

$$h_{02} = -\frac{\omega R}{c}\left[\left(1+\frac{z'}{R}\right)\sin\theta_0 + \frac{x'}{R}\cos\theta_0\right] + \frac{a r_s}{R^2}\left[\left(1-\frac{2z'}{R}\right)\sin\theta_0 + \frac{x'}{R}\cos\theta_0\right], \quad (27)$$

$$h_{03} = \left(\frac{\omega\, R}{c} - \frac{a r_s}{R^2}\right)\frac{y'}{R}\sin\theta_0, \quad (28)$$

$$h_{11} = -\frac{a^2}{R^2}\left[\left(1-\frac{2z'}{R}\right)\cos^2\theta_0 - \frac{x'}{R}\sin(2\theta_0)\right], \quad (29)$$

$$h_{12} = \frac{1}{2}\frac{a^2}{R^2}\frac{y'}{R}\sin(2\theta_0), \quad (30)$$

$$h_{13} = -\left(\frac{r_s}{R} - \frac{a^2}{R^2}\right)\frac{x'}{R}, \quad (31)$$

$$h_{22} = -\frac{a^2}{R^2}\left(1-\frac{2z'}{R}\right), \quad (32)$$

$$h_{23} = -\left(\frac{r_s}{R} - \frac{a^2}{R^2}(1+\sin^2\theta_0)\right)\frac{y'}{R}, \quad (33)$$

$$h_{33} = -\frac{r_s}{R}\left(1-\frac{z'}{R}\right) + \frac{a^2}{R^2}\left[\left(1-\frac{2z'}{R}\right)\sin^2\theta_0 + \frac{x'}{R}\sin(2\theta_0)\right]. \quad (34)$$

5 Relativistic derivation of the COW and Sagnac effects

We are now in a position to give a relativistic account of the COW and neutron Sagnac effects. To do so, we need a relativistic expression for the phase shift, which comes from the Feynman–Dirac formula $\exp(iS/\hbar)$, the amplitude for a particle to travel along a path, with $S = \int \mathcal{L}\, dt$ being the action along the path. The relativistic expression for S is $-mc\int ds$, with $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2 d\tau^2$, τ being proper time. Dividing the expression for ds^2 by ds gives

$$ds = g_{\mu\nu}\frac{dx^\mu}{ds}dx^\nu = \frac{1}{c}g_{\mu\nu}\frac{dx^\mu}{d\tau}dx^\nu, \quad (35)$$

so

$$S = -m\int g_{\mu\nu}\frac{dx^\mu}{d\tau}dx^\nu = -\int g_{\mu\nu}p^\mu dx^\nu = -\int p_\mu dx^\mu, \quad (36)$$

consistent with Eq. (2) above.

We may now proceed, following Refs. [10,11], by stating that the phase Φ_{AB} accumulated by a particle moving from spacetime event A to event B is, invoking the

weak field approximation,

$$\Phi_{AB} = -\frac{mc}{\hbar}\int_A^B ds \approx -\frac{2\pi}{\lambda_{\mathrm{C}}}\int_A^B \left(ds_{\mathcal{M}} + \frac{1}{2}h_{\mu\nu}\frac{dx^\mu}{ds_{\mathcal{M}}}dx^\nu\right), \tag{37}$$

where $h_{\mu\nu}$ is the deviation of the metric $g_{\mu\nu}$ from its Minkowskian form $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, with $ds^2_{\mathcal{M}} = \eta_{\rho\sigma}dx^\rho dx^\sigma$, and $\lambda_{\mathrm{C}}/(2\pi) = \hbar/(mc)$, with λ_{C} being neutron's Compton wavelength. Eq. (37) represents the action, normalized to Planck's constant, of a *freely falling* gravitational probe. We assume, as an additional hypothesis, that (37) can also be applied to a probe whose worldline is shaped by, say, a collection of ideally reflecting mirrors that are at rest relative to the chosen coordinate system. (A mirror is regarded as ideal if on reflection there is no change of particle's energy and of the tangential component of its momentum, while the normal component of the momentum changes sign.) A similar assumption for calculating gravitational effects, though in a different context, was made in Ref. [21]. The *gravitationally induced* phase is then given by

$$\Phi_{AB} = -\frac{\pi}{\lambda_{\mathrm{C}}}\int_A^B h_{\mu\nu}u^\mu_{\mathcal{M}}dx^\nu, \tag{38}$$

where $u^\mu_{\mathcal{M}} = dx^\mu/ds_{\mathcal{M}} = (\gamma, \gamma\mathbf{v}/c)$ is the usual relativistic four-velocity of the particle, $\mathbf{v}$ is its three-velocity, and γ is the corresponding gamma-factor. The phase difference between the two interfering paths is then

$$\Delta\Phi = -\frac{\pi}{\lambda_{\mathrm{C}}}\oint h_{\mu\nu}u^\mu_{\mathcal{M}}dx^\nu, \tag{39}$$

where the line integral is taken around the loop formed by the paths.

We now make an important observation that, in the linearized approximation, $\sim \mathcal{O}(h_{\mu\nu})$, used in Eq. (38), neutron's speed, $v \equiv |\mathbf{v}|$, should be treated as *constant*. Any change in the speed acquired due to gravity, etc., had already been taken into account when we made the linearized approximation (38). Thus, the gravitationally induced phase difference between the interfering paths may be found from the formula

$$\Delta\Phi = -\frac{\pi\gamma c}{\lambda_{\mathrm{C}}}\oint\left[\left(h_{00} + \frac{h_{i0}v^i}{c}\right)dt + \left(h_{0j} + \frac{h_{ij}v^i}{c}\right)\frac{dx^j}{c}\right], \tag{40}$$

where $v = \sqrt{\delta_{ij}v^iv^j}$ is regarded as constant.

Equation (40) represents the accumulated phase difference for a *single* orientation of the loop. This phase difference, which we call intrinsic, is not directly observable. In an actual experiment, at least *two* orientations are involved, and it is the *shift* in the intrinsic phase difference during the rotation of the loop from one position to the other that is experimentally measurable.

Assuming that the loop is a *rectangle* placed in the $x'y'$-plane, with the sides parallel to the x' and y' axes, we have $z' = 0$ and $\mathbf{v} = (v_x, v_y, 0)$, and upon using (40), find the corresponding intrinsic phase difference,

$$
\begin{aligned}
(\Delta\Phi)_{x'y'} &= -\frac{\pi\gamma c}{\lambda_C}\left\{\int_{(0,0,0)}^{(\Delta x',0,0)} - \int_{(0,\Delta y',0)}^{(\Delta x',\Delta y',0)}\right\}\left[\left(h_{00}+\frac{h_{10}v}{c}\right)dt + \left(h_{01}+\frac{h_{11}v}{c}\right)\frac{dx'}{c}\right] \\
&\quad -\frac{\pi\gamma c}{\lambda_C}\left\{\int_{(\Delta x',0,0)}^{(\Delta x',\Delta y',0)} - \int_{(0,0,0)}^{(0,\Delta y',0)}\right\}\left[\left(h_{00}+\frac{h_{20}v}{c}\right)dt + \left(h_{02}+\frac{h_{22}v}{c}\right)\frac{dy'}{c}\right] \\
&= -\frac{\pi\gamma c}{\lambda_C}\left\{\int_{(0,0,0)}^{(\Delta x',0,0)} - \int_{(0,\Delta y',0)}^{(\Delta x',\Delta y',0)}\right\}\left(\frac{h_{00}}{v}+\frac{2h_{10}}{c}+\frac{h_{11}v}{c^2}\right)dx' \\
&\quad -\frac{\pi\gamma c}{\lambda_C}\left\{\int_{(\Delta x',0,0)}^{(\Delta x',\Delta y',0)} - \int_{(0,0,0)}^{(0,\Delta y',0)}\right\}\left(\frac{h_{00}}{v}+\frac{2h_{20}}{c}+\frac{h_{22}v}{c^2}\right)dy' \\
&= +\frac{\pi\gamma c}{\lambda_C v}\left\{\frac{4v}{c}\left(\frac{\omega R}{c}-\frac{ar_s}{R^2}\right)\cos\theta_0 + \frac{\omega^2R^2}{c^2}\sin(2\theta_0)\right\}\frac{\Delta x'\Delta y'}{R},
\end{aligned}
\tag{41}
$$

which vanishes in the $a, \omega \to 0$ limit, as had to be expected. In a similar manner, for the $z'x'$ and $y'z'$ orientations, we get

$$
\begin{aligned}
(\Delta\Phi)_{z'x'} &= \frac{\pi\gamma c}{\lambda_C v}\left[\frac{r_s}{R} - \frac{\omega^2R^2}{c^2}\left(2\sin^2\theta_0 - \sin(2\theta_0)\right)\right. \\
&\quad \left. + \frac{v^2}{c^2}\frac{a^2}{R^2}\left(2\cos^2\theta_0 - \sin(2\theta_0)\right)\right]\frac{\Delta z'\Delta x'}{R},
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
(\Delta\Phi)_{y'z'} &= \frac{\pi\gamma c}{\lambda_C v}\left[\frac{r_s}{R} - \frac{2\omega^2R^2}{c^2}\sin^2\theta_0 - \frac{2v}{c}\left(\frac{2\omega R}{c}+\frac{ar_s}{R^2}\right)\sin\theta_0\right. \\
&\quad \left. + \frac{v^2}{c^2}\frac{2a^2}{R^2}\right]\frac{\Delta y'\Delta z'}{R}.
\end{aligned}
\tag{43}
$$

Combining Eqs. (41) and (42), and assuming that the loop is now rotated around the x'-axis from horizontal $x'y'$ to vertical $z'x'$ position, we get, using $\Delta x' \equiv L$ and $\Delta y' = \Delta z' \equiv H$, the experimentally observable COW change of phase,

$$
\begin{aligned}
(\Delta\Phi)_{\mathrm{COW}} &\equiv (\Delta\Phi)_{z'x'} - (\Delta\Phi)_{x'y'} \\
&= \frac{\pi\gamma c}{\lambda_C v}\frac{LH}{R}\left\{\frac{r_s}{R} - \frac{4v}{c}\left(\frac{\omega R}{c}-\frac{ar_s}{R^2}\right)\cos\theta_0 - \frac{2\omega^2R^2}{c^2}\sin^2\theta_0\right. \\
&\quad \left. + \frac{v^2}{c^2}\frac{a^2}{R^2}\left(2\cos^2\theta_0 - \sin(2\theta_0)\right)\right\} \\
&= \gamma\frac{mgA}{\hbar v} + (\text{terms in } \omega \text{ and } \omega^2),
\end{aligned}
\tag{44}
$$

where we have made the identification

$$
\frac{r_s}{R^2} \equiv \frac{2g}{c^2},
\tag{45}
$$

with g being the acceleration due to gravity at the earth's surface. It therefore turns out that in this relativistic formulation the COW phase shift is merely the simple result (7), corrected by the factor γ, and beyond that, further corrected (slightly surprisingly!)

by terms resulting from the rotation of the earth. These terms are two or more orders of magnitude smaller than the first terms in (44): $r_s/R = 1.4 \times 10^{-9}$, $4v\omega R/c^2 = 4.7 \times 10^{-11}$, $\omega R/c = 1.6 \times 10^{-6}$, $ar_s/R^2 = 0.9 \times 10^{-15}$, where we have taken $v = 2.2 \times 10^3$ m/s for thermal neutrons.

On the other hand, combining Eqs. (42) and (43), and assuming that the loop is rotated around the z'-axis from vertical $z'x'$ to vertical $y'z'$ position, we get, using $\Delta x' = \Delta y' \equiv L$ and $\Delta z' \equiv H$, the phase shift

$$\begin{aligned}(\Delta\Phi)_{\text{Sagnac}} &\equiv (\Delta\alpha)_{y'z'} - (\Delta\alpha)_{z'x'} \\ &= +\frac{\pi\gamma c}{\lambda_C v}\frac{LH}{R}\left\{-\frac{2v}{c}\left(\frac{2\omega R}{c}+\frac{ar_s}{R^2}\right)\sin\theta_0 - \frac{\omega^2R^2}{c^2}\sin(2\theta_0)\right. \\ &\quad \left. +\frac{v^2}{c^2}\frac{a^2}{R^2}\left(2\sin^2\theta_0+\sin(2\theta_0)\right)\right\} \\ &= -\gamma\frac{2m\omega A}{\hbar} + (\text{terms in } a, a^2, \omega^2). \end{aligned} \tag{46}$$

We see, similarly to the COW case, that the magnitude of the Sagnac effect is the same as obtained in the simple derivation, corrected by γ, and modified by considerably smaller terms.

6 Summary

We conclude that by making the weak field approximation we may straightforwardly derive expressions for the COW and neutron Sagnac phase shifts. Our general relativistic calculation yields the same results as simple semi-classical arguments do, corrected only by the relativistic factor γ, and by higher order terms involving the angular velocity of the earth.

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